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Efficient Specialization in Ricardian Production

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Wolfram F. Richter and Joachim Rosenmüller[1]

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Abstract

It is well known that the analysis of efficient specialization in Ricardian production with many countries and many commodities cannot be broken down to the simple case of two countries and two commodities. By drawing on some recent results of convex geometry and the theory of cephoids, this paper characterizes efficient patterns of incomplete specialization in the general case.

JEL Classification: F10

Keywords: Ricardian trade; efficient specialization; comparative cost; cephoids; deGua simplexes

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1. Introduction

Ricardo (1817) referred to the framework of two countries and two commodities when studying the notions of comparative advantage and efficient specialization in production. For many years however it remained totally unclear how representative the Ricardian analysis is for the general case of many countries and many commodities. The investigation of the general case was initiated by Graham (1948) by the use of some numerical examples. McKenzie (1954) then showed a way of systematically exploring the general case by drawing on activity analysis. Jones (1961) used this approach and developed necessary and sufficient conditions for efficient multilateral specialization. However, he only managed to characterize those efficient patterns of specialization in which each country is completely specialized. Such a pattern of complete specialization requires each country to be assigned to the production of just one particular commodity. Jones (1961) proved that a pattern of complete specialization is efficient if, and only if, the associated assignment minimizes the product of labour requirements in an associated class of competing assignments. Thus he demonstrated that the criterion of the efficient assignment involves minimizing the product of labour coefficients involved in the assignment. In particular, a comparison of bilateral ratios of labour coefficients ("comparative costs") is not enough. Hence, the general case cannot be reduced to the study of two countries and two commodities. See also Jones (2005).

The result marks a major break through in the study of the many-country, many-commodity case. Still, it does not present a complete theory of efficient specialization. This is so, as the result only allows one to characterize efficient patterns of complete specialization. It cannot be used to characterize efficient patterns of incomplete specialization which have been the object of some most tedious calculations of Graham (1948). The present paper closes this gap. It does so by focussing on those efficient patterns of specialization which display some minimal degree of specialization. What we mean by such minimal degree of specialization is best explained with reference to Ricardo's case of two countries and two commodities. A pattern of specialization is complete if each country produces just one commodity. An efficient pattern displays a minimal degree of specialization if one country is specialized in production while the other country is not. The minimal degree cannot be zero in the sense that no country is specialized in production if the countries differ in comparative costs. Ruling out such cases of degeneracy, this paper provides a complete characterization of efficient patterns of minimal specialization for the many-country, many-commodity case. The characterization makes use of some recent results of convex geometry and the theory of cephoids. *Inter alia*, it is shown that the rule of Jones suggesting to minimize products of labour requirements has to be generalized to cover cases of incomplete specialization. The generalized rule requires maximizing the product of values of labour units over an associated class of specializations. The differences in the criteria will be clarified in the detail below. Here it suffices to stress the most striking difference. The product of labour units relies on prices as a weighting scheme while the product of labour requirements can be stated without reference to prices.

The paper is organized as follows. Section 2 sets up the standard model of Ricardian production. In Section 3 the notion of non-degenerate production is defined. Section 4 provides results and Section 5 summarizes.

2. Efficient production

Consider $K \geq 2$ countries and $n \geq 2$ commodities. Each country is endowed with $L_k > 0$ units of labour. The endowment is interpreted as exogenous labour supply. Let $a_i^k > 0$ be the number of labour units required in country k to produce one unit of commodity $i \in \{1,..,n\}$. The quantities of i are denoted by $x_i \geq 0$. We make use of the following notation: $x = (x_1,..,x_n)$, $a^k = (a_1^k,..,a_n^k)$ and $a^k \cdot x \equiv \sum_{i=1}^{n} a_i^k x_i$. A production plan x is said to be *feasible* for country k if the demand for labour does not exceed the supply, $a^k \cdot x \leq L_k$. Let Π^k denote the set of feasible production plans of country k. The set of production plans which is feasible for the world is defined by the sum of the sets Π^k:

$$\Pi^{\bullet} \equiv \sum_{k=1}^{K} \Pi^k = \{X = x^1 + ... + x^K \,|\, x^1 \in \Pi^1,..,x^K \in \Pi^K\}.$$

Let $p \in \mathbf{R}_{++}^n$ be a vector of strictly positive commodity prices to be interpreted as prices ruling on world markets. Consider a world production plan $\bar{X} \in \Pi^{\bullet}$ holding

$$p \cdot \bar{X} = \max \{ p \cdot X \,|\, X \in \Pi^{\bullet} \}.$$

We then say that $\bar{X}$ is an *efficient* production plan and that p is *supporting* $\bar{X}$. Obviously, if p is supporting the plan $\bar{X}=\sum_{k=1}^{K}\bar{x}^k$ with $\bar{x}^k \in \Pi^k$, then p is also supporting each individual country plan $\bar{x}^k$. I.e. $p\cdot\bar{x}^k = \max\{p\cdot x^k \,|\, x^k \in \Pi^k\}$.

Let us call p_i/a_i^k the *value of a unit of labour*, or *the value of labour unit* for short, when labour is used in country k in the production of commodity i. Set

$$w_k = w_k(p) \equiv \max_{i=1,..,n} p_i/a_i^k \tag{1}$$

to be interpreted as the wage rate of country k supported by world prices p. By definition, the wage rate of country k equals the maximal value of a unit of labour that can be earned in country k when varying the use of labour across sectors. The equality of $w_k = p_i/a_i^k$ is interpreted as a *zero-profit condition*. Obviously, it does not pay to produce commodity i if $w_k > p_i/a_i^k$. In other words, if $\bar{X}=\sum_{k=1}^{K}\bar{x}^k$ is supported by p and if $w_k > p_i/a_i^k$, then $\bar{x}_i^k$=0.

The proof is straightforward: Just assume $\bar{x}_{j_0}^k$>0, $w_k > p_{j_0}/a_{j_0}^k$, and $w_k = p_{j_1}/a_{j_1}^k$. Then one can define the production plan $\tilde{x}$ as follows:

$$\tilde{x}_i^k \equiv \begin{cases} \bar{x}_i^k & i \neq j_0, j_1 \\ 0 & i = j_0 \\ \bar{x}_{j_1}^k + a_{j_0}^k \bar{x}_{j_0}^k / a_{j_1}^k & i = j_1 \end{cases}$$

This results in

$p\cdot\tilde{x}^k - p\cdot\bar{x}^k = p_{j_1}\tilde{x}_{j_1}^k - p_{j_0}\bar{x}_{j_0}^k - p_{j_1}\bar{x}_{j_1}^k = p_{j_1}[\bar{x}_{j_1}^k + a_{j_0}^k\bar{x}_{j_0}^k/a_{j_1}^k] - p_{j_0}\bar{x}_{j_0}^k - p_{j_1}\bar{x}_{j_1}^k = (p_{j_1}a_{j_0}^k/a_{j_1}^k - p_{j_0})\bar{x}_{j_0}^k = (w_k a_{j_0}^k - p_{j_0})\bar{x}_{j_0}^k > 0$. This is a contradiction to the assumed efficiency of $\bar{x}_i^k$.

These observations give rise to the following definition. For given price vector p, let

$$I_k(p) \equiv \{i=1,..,n \,|\, w_k = p_i/a_i^k\}$$

denote the set of commodities which can be produced in country k in positive amounts without making a loss. If $\bar{X}=\sum_k \bar{x}^k$ is supported by p and if $i \notin I_k(p)$, then $\bar{x}_i^k$=0. The set $I_k(p)$ is called country k's *profile of specialization* (in production) *supported by p*. A family of such specialization profiles $\Im(p)=(I_1(p),...,I_K(p))$ is termed a *pattern of specialization* *supported by p*. Our objective is to study the set of patterns of minimal specialization that can be supported by world prices.

There is a competing approach which does not focus on patterns of specialization but on assignments. This competing approach has been pursued by Jones (1961). According to Jones, an *assignment* specifies for every commodity i all those countries k which can produce i without making a loss. Hence an assignment is a vector of index sets $(K_1(w),..,K_n(w))$ with

$$K_i(w) \equiv \{\bar{k} = 1,..,K \,\Big|\, p_i \equiv a_i^{\bar{k}} w_{\bar{k}} = \min_{k=1,..,K} a_i^k w_k\}.$$

The approach of Jones (1961) relies on characterizing those assignments in an associated class of assignments which determine the world production frontier. If the focus is on patterns of complete specialization with $K=n$, it does not make much difference whether one studies assignments or patterns of specialization. There is a 1-1 relationship. This is different in the general case. We choose to study patterns of specialization because this allows us to draw more directly on some recent results of convex geometry.

3. Non-degeneracy

Let us start by having a closer look at patterns of complete specialization when there are just as many countries as there are commodities, $K=n$. Fix p and let $i_p(k)$ be the uniquely determined commodity produced by country k without making a loss, $I_k(p) = \{i_p(k)\}$. Jones (1961) recognized that one can set up some problem of minimization so that i_p is the solution. The objective function to be minimized is the product of labour coefficients. To see this, let $\pi \circ i_p$ be the pattern of complete specialization resulting when permuting the indices $i_p(1),...,i_p(K)$. By the definition of w_k, i_p,

$$w_k a_{i_p(k)}^k = p_{i_p(k)} \text{ and } w_k a_{\pi\circ i_p(k)}^k \geq p_{\pi\circ i_p(k)} \text{ for all } k.$$

Hence

$$\prod_{k=1}^K w_k \prod_{k=1}^K a_{i_p(k)}^k = \prod_{k=1}^K p_{i_p(k)} = \prod_{k=1}^K p_{\pi\circ i_p(k)} \leq \prod_{k=1}^K w_k \prod_{k=1}^K a_{\pi\circ i_p(k)}^k \Rightarrow$$

$$\prod_{k=1}^K a_{i_p(k)}^k = \min_\pi \prod_{k=1}^K a_{\pi\circ i_p(k)}^k. \qquad (2)$$

Eq (2) demonstrates that a pattern of complete specialization can be supported by prices if, and only if, it can be viewed as the solution of an appropriately specified minimization.

Clearly, the pattern of complete specialization supported by the given price vector p is unique only if the minimization (2) has a unique solution. *A priori*, this need not be the case. This is well known and the study of such degenerate scenarios has long plaid a prominent role in the literature. Most prominent is

Ricardo's case of degeneracy: $a_1^1 / a_2^1 = a_1^2 / a_2^2 \Leftrightarrow a_1^1 a_2^2 = a_2^1 a_1^2$ (3)

Condition (3) is obviously compatible with (2) when setting K=n=2, $i_p(k) = k$ (k=1,2) and $\pi \circ i_p(1) = 2$, $\pi \circ i_p(2) = 1$. Ricardo interprets (3) as a condition of equality in comparative costs. In the case of such equality, trade fails to be gainful. Nothing is gained by letting the production of any commodity be concentrated in only one single countries. It is interesting to note that the question of whether trade is gainful in the more general case with K=n>2 cannot be broken down to the question of whether trade is gainful in 2×2 scenarios. To see this, inspect

Jones' case of degeneracy: $a_1^1 a_2^2 a_3^3 = a_2^1 a_3^2 a_1^3$. (4)

Condition (4) is compatible with (2) when setting K=n=3, $i_p(k) = k$ (k=1,2,3) and $\pi \circ i_p(k) = k+1 \mod 3$. Clearly, (4) does not exclude the constellation with $a_1^1 a_2^2 < a_2^1 a_1^2$, $a_2^2 a_3^3 < a_3^2 a_2^3$, and $a_1^1 a_3^3 < a_3^1 a_1^3$. In purely bilateral comparisons each country k then has a comparative cost advantage in the production of k. Still, trade is not gainful. The pattern of specialization in which country 1 produces commodity 2, country 2 commodity 3, and country 3 commodity 1 is supported by the same price vector as the pattern of specialization in which each country k produces commodity k.

Scenarios like (2) or (3) are rightly considered to be degenerated. By slight perturbation of the labour requirements the equations turn into inequalities. This observation however raises the question of how to specify the notion of degeneracy in the fully general case with $K \neq n$. To answer this question, fix some arbitrary pattern of specialization, $\mathfrak{I} = (I_1, ..., I_K)$, ($I_k \subseteq \{1, ..., n\}, 1 \le k \le K$), and set

$$\mathrm{E}(\mathfrak{I}) \equiv \{(w, p) \in \mathbf{R}_{++}^{K+n} \,\big|\, p_i = w_k a_i^k \text{ for } i \in I_k, 1 \le k \le K\}.$$

In economic terms $\mathrm{E}(\mathfrak{I})$ defines a set of competitive equilibria in prices and wage rates associated with the vectors of labour requirements $(a^1, ..., a^K)$. In mathematical terms $\mathrm{E}(\mathfrak{I})$ defines a set of solutions to a set of homogeneous linear equations. Obviously, if $\mathrm{E}(\mathfrak{I})$ is non-empty, it has a vector space dimension, *dim* E, which is not smaller than one. This is so as zero-profit conditions can be multiplied through by any positive real number without destroying the equations. The set $\mathrm{E}(\Im)$ is a subset of $\mathbf{R}^{n+K}$ constrained by $\sum_{k=1}^{K}|I_k|$ zero-profit conditions where $|I_k|$ counts the number of elements in I_k. Hence, *dim* $\mathrm{E}(\Im)$ $\geq n+K-\sum_{k=1}^{K}|I_k|$.

Definition: The family of vectors of labour requirements $(a^1,...,a^K)$ is called *non-degenerate* if for any pattern of specialization $\Im$ the resulting set of zero-profit conditions is non-degenerate,

$$dim\ \mathrm{E}(\Im) = \max\ \{0, n+K-\sum_{k=1}^{K}|I_k|\}\ . \tag{5}$$

It is straightforward to show that this general notion of (non-) degeneracy is fully compatible with the notions defined by (3) and (4). To see this, consider Ricardo's case first.

Hence suppose that the set $\mathrm{E}(\Im)$ with $\Im = (I_1 = \{1,2\}, I_2 = \{1,2\})$ is non-empty. Then non-zero vectors w and p exist solving the system of zero-profit conditions,

$$\begin{array}{ll} w_1 a_1^1 = p_1 & w_2 a_2^2 = p_2 \\ w_1 a_2^1 = p_2 & w_2 a_1^2 = p_1 \end{array}.$$

This system is compactly written in matrix form:

$$A\begin{pmatrix} w \\ p \end{pmatrix} = 0 \quad \text{with } A \equiv \begin{pmatrix} a_1^1 & 0 & -1 & 0 \\ 0 & a_2^2 & 0 & -1 \\ a_2^1 & 0 & 0 & -1 \\ 0 & a_1^2 & -1 & 0 \end{pmatrix}.$$

The existence of non-trivial solutions (w,p) implies $0 = \det A = a_1^1 a_2^2 - a_2^1 a_1^2$ so that (3) holds and equally condition (5) with strict inequality, *dim* $\mathrm{E}(\Im) \geq 1 > 0 = 2+2-2-2 = n+K-\sum_{k=1}^{K}|I_k|$.

Vice versa, suppose that $n+K-\sum_{k=1}^{K}|I_k|$ = *dim* $\mathrm{E}(\Im)$ > 0. Then $\sum_{k=1}^{K}|I_k|$ < 4 and $\Im \neq (I_1 = \{1,2\}, I_2 = \{1,2\})$. Hence (3) cannot hold.

In the case of Jones, suppose that the set $\mathrm{E}(\mathfrak{I})$ with $\mathfrak{I} = (I_1 = \{1,2\}, I_2 = \{2,3\}, I_3 = \{3,1\})$ is non-empty. Then vectors *w* and *p* exist solving the system of equations,

$$\begin{matrix} w_1 a_1^1 = p_1 & w_2 a_2^2 = p_2 & w_3 a_3^3 = p_3 \\ w_1 a_2^1 = p_2 & w_2 a_3^2 = p_3 & w_3 a_1^3 = p_1 \end{matrix}.$$

Again, this system can be written in matrix form, $A\begin{pmatrix} w \\ p \end{pmatrix} = 0$. It is straightforward to verify $\det A = a_2^1 a_3^2 a_1^3 - a_1^1 a_2^2 a_3^3$. The existence of non-trivial solutions implies $0 = \det A$ so that (4) holds and equally condition (5) with strict inequality, *dim* $\mathrm{E}(\mathfrak{I}) \geq 1 > 0 = 3+3-2-2-2 = n + K - \sum_{k=1}^{K} |I_k|$.

4. The efficiency of patterns of minimal specialization

The first result we are going to prove allows us to classify the set of efficient patterns of minimal specialization. The classification makes use of the notion of a reference vector. This notion has been introduced by Pallaschke and Rosenmüller, 2007a. Fix some arbitrary pattern of specialization, $\mathfrak{I} = (I_1,...,I_K)$. The vector of integers $r = (r_1,..,r_K) \in \mathbb{N}^K$ the *k*'th entry of which counts the number of indices in I_k, $r_k = |I_k|$, is called *reference vector* for $\mathfrak{I}$. We also say that $\mathfrak{I}$ is *associated with r*. Let us call the reference vector *r maximal* if

$$\sum_{k=1}^{K} r_k = n + K - 1. \tag{6}$$

The term "maximal" is justified by the following observation. Suppose there is a pattern of specialization $\mathfrak{I}$ with an associated reference vector *r* so that $\sum_{k=1}^{K} r_k$ exceeds $n + K - 1$. Then $0 \geq n + K - \sum_{k=1}^{K} r_k$. Because of (5), either $\mathrm{E}(\mathfrak{I})$ would have to be empty or $(a^1,...,a^K)$ would have to be degenerate. Ruling out such cases justifies to call *r* maximal whenever (6) holds.

Definition: A *pattern of specialization* $\mathfrak{I} = (I_1,...,I_K)$ is called

- *minimal* if the associated reference vector *r* is maximal;
- *efficient* if prices *p* exist supporting this pattern, $\mathfrak{I} = \mathfrak{I}(p)$.

A pattern of complete specialization is obviously associated with the reference vector $r=(1,..,1)$. Assuming non-degeneracy, a pattern of specialization can therefore be efficient only if the associated reference vector holds $K \le \sum_{k=1}^{K} r_k \le n+K-1$.

Theorem: Assume non-degeneracy of $(a^1,...,a^K)$. For every maximal reference vector r there exists exactly one efficient pattern of (minimal) specialization $\Im = \Im(r)$ such that r is the associated reference vector.

Proof: Geometrically speaking, country k's set of feasible production plans Π^k is a simplex spanned by the origin and scalar multiples of the unit vectors. In convex geometry this is termed a *DeGua simplex*. The finite sum of such DeGua simplexes, $\Pi^\bullet = \sum_{k=1}^{K} \Pi^k$, is named a *cephoid* by Pallaschke and Rosenmüller (2010). Such cephoids are constrained by supporting hyperplanes of dimension n-1. The intersections of such hyperplanes with $\Pi^\bullet$ are termed *facets*. Such facets are called *maximal* if their dimension is n-1. Pallaschke and Rosenmüller (2007a) prove that for each maximal reference vector r there exists a unique maximal facet F=F(r) of $\Pi^\bullet$ described by a pattern of specialization $\bar{\Im} = (\bar{I}_1,...,\bar{I}_K)$ associated with r such that

$$F = \sum_{k=1}^{K} \Delta^k_{\bar{I}_k} \quad \text{with} \quad \Delta^k_{\bar{I}_k} \equiv \{x \in \Pi^k \,|\, x^k_i = 0 \ \text{ for } i \notin \bar{I}_k\} \ .$$

Let p=p(r) be a vector supporting F= F(r). As F is maximal, p is necessarily unique up to normalization. Define $w_k(p)$ and $I_k(p)$ as before. It suffices to prove $I_k(p) = \bar{I}_k$ for all k=1,..,K:

Consider some $i \in \bar{I}_k$. By definition, there exists a world production plan $\bar{X} = \sum_{k'=1}^{K} \bar{x}^{k'} \in F$ with $\bar{x}^k \in \Delta^k_{\bar{I}_k}$ and $\bar{x}^k_i \neq 0$. As p is supporting F, p is equally supporting $\bar{X}$. If commodity i were no element of $I_k(p)$, $\bar{x}^k_i$ would have to equal zero. As this would be contradictory, $\bar{I}_k$ must be a subset of $I_k(p)$.

Consider some $i \in I_k(p)$ next. By definition, $p_i = w_k a_i^k$. Fix some $\overline{X} = \sum_{k=1}^{K} \overline{x}^k$ with $\overline{x}_i^k > 0$ which can be supported by p. Hence, $\overline{X} = \sum_{k=1}^{K} \overline{x}^k \in F = \sum_{k=1}^{K} \Delta_{\overline{I}_k}^k$. If i were no element of $\overline{I}_k$, $\overline{x}_i^k$ would have to equal zero and which would be contradictory.

An elaborate example of Ricardian production with K=n=10 can be found in Graham (1948, Chap. VI). In Fn. 6 on page 95 Graham concedes that it has been "a tedious process of trial and error" through which he eventually succeeded to determine the pattern of specialization that can be supported by some given vector of prices. The solution is such that the associated reference vector $(4,4,3,2,1,\ldots,1) \in \mathbb{N}^{10}$ satisfies the constraint (6). The example presumably stands for the most ambitious attempt ever undertaken to determine a non-trivial efficient pattern of minimal specialization just by hand.[1]

The above Theorem only ensures the existence of efficient patterns of specialization. It cannot be interpreted as a prescription for constructing particular ones. However, efficient patterns of specialization display characteristic features which can be used to check whether a particular pattern is efficient or not. The first characteristic feature is the following. Efficiency is only given if the product of values of labour units is maximized by the pattern under consideration over an associated class of specializations. Before proving this statement, the notion of an associated class of specializations has to be specified.

Definition: The set of all patterns of specializations which are associated with a given reference vector r is called the *class of specializations associated with r*.

Fix some maximal reference vector r and assume non-degeneracy of $(a^1,\ldots,a^K)$. Let $\Im(r)$ be the efficient pattern of minimal specialization which according to the above Theorem is associated with r. As the pattern is efficient, prices $p = p(r)$ exist so that $\Im(r)$ is supported by these prices, $\Im(r) = \Im(p) = (I_1(p),\ldots,I_K(p))$. Hence, $p_i / a_i^k = w_k \geq p_j / a_j^k$ must hold for all $i \in I^k(p)$ and $j \in \{1,..,n\}$. Writing

[1] Patterns of minimal specialization are considered to be trivial if one country produces all commodities and all other countries produce just one commodity.

$$\prod_{\Im}(p_i / a_i^k) \equiv \prod_{i_1 \in I^1}(p_{i_1} / a_{i_1}^1) \cdot \ldots \cdot \prod_{i_K \in I^K}(p_{i_K} / a_{i_K}^K)$$

this implies:

$$\prod_{\Im(r)}(p_i(r)/a_i^k) = w_1^{r_1} \cdot \ldots \cdot w_K^{r_K} \geq \prod_{\Im}(p_i(r)/a_i^k) \text{ for all } \Im = (I_1,\ldots,I_K) \text{ which} \tag{7}$$

are elements of the class of specializations associated with *r*.

Remember that $\Im(r)$ is uniquely determined. Some $\overline{\Im}$ is therefore efficient if, and only if $\overline{\Im}$ maximizes $\prod_{\Im}(p_i(r)/a_i^k)$ in the class of specializations associated with *r*. This can be expressed less technically when remembering that p_j / a_j^k can be interpreted as the value of a unit of labour in the production of *j*. Call $\prod_{\Im}(p_i(r)/a_i^k)$ the *product of values of labour units associated with* $\Im$ *and r*.

Corollary: Assume non-degeneracy of $(a^1,\ldots,a^K)$ and some maximal reference vector *r*. Let $\overline{\Im}$ be a pattern of specialization in the class of specializations associated with *r*. This pattern is efficient if, and only if, it maximizes the associated product of values of labour units in this class of specializations.

The Corollary strongly reminds one of the result derived by Jones (1961). However, there are some notable differences. Jones focuses on complete specializations which require each country to be assigned to the production of just one particular commodity. He then proves that a pattern of complete specialization is efficient if, and only if, it minimizes the product of labour requirements in the class of complete specializations obtained when permuting the assignment of countries to commodities (see (2), above). The Corollary deviates from this characterization in two relevant respects. First, labour requirements are substituted by values of labour units. This means that prices are used as a weighting scheme when optimizing. Secondly, the classes over which optimization occurs is defined differently. Just compare condition (7) with (2). While the minimization in (2) is over all permutations of some assignment $i_p(1),\ldots,i_p(K)$, the maximization in (7) is over all patterns of specialization $\Im = (I_1,\ldots,I_K)$ with $\left|I^k\right| = r_k$ for *k*=1,..,*K*. The latter set of specializations would even be larger if we were to apply (7) to the case of complete specialization. The set of all

$\Im = (I_1,...,I_K)$ with $|I^k| = 1$ contains elements that are not necessarily permutations of some given $i_p(1),...,i_p(K)$.

There are additional characteristic features shared by efficient patterns of minimal specialization. Let $\mathrm{J}(\Im) \subseteq \{1,..,n\}$ be the set of indices picking all those commodities which are produced *jointly* by more than one country.

Proposition: Assume non-degeneracy of $(a^1,...,a^K)$. For each efficient pattern of minimal specialization $\overline{\Im} = (\overline{I}_1,...,\overline{I}_K)$ the following statements hold true:

(i) There are at most $K-1$ commodities being produced in two or more countries. I.e., $|\mathrm{J}(\overline{\Im})| \leq K-1$.

(ii) Each country produces at least one commodity jointly with another country. I.e., $|\overline{I}_k \cap \mathrm{J}(\overline{\Im})| \geq 1$ for all k=1,..,K.

The proof of the Proposition follows again from Pallaschke and Rosenmüller (2010). More precisely, the Proposition is a straightforward application of the second item listed in the "Coincidence Theorem" proved by Pallaschke and Rosenmüller.

In order to illustrate the Proposition, consider the case of K=3 and n=4. Statement (i) says that at most two of the four commodities are produced jointly by the three countries. Obviously, the two patterns of specialization,

$$\Im^1 = (I_1^1 = \{1,3,4\}, I_2^1 = \{1,2\}, I_3^1 = \{3\}) \text{ and } \Im^2 = \Im^2 = (I_1^2 = \{1,2,4\}, I_2^2 = \{1,2\}, I_3^2 = \{3\})$$

both pass this test. In the first case, the set of jointly produced commodities is J^1=$\{1,3\}$ and in the second case the set is J^2=$\{1,2\}$. Both sets do not contain more than two elements. However, only the first example passes test (ii). In the second case, the set $I_3^2 \cap \mathrm{J}^2$=$\{3\} \cap \{1,2\}$ is void so that (ii) is violated.

The example of Graham (1948, Chap. VI) for K=n=10 being cited above is such that seven commodities are produced in two countries and another commodity is even produced in three countries. Furthermore, there are three countries for which the number of commodities jointly produced with another country is three, $|I_k \cap \mathrm{J}|$=3.

In the case of some efficient and minimal specialization, relative wages are uniquely determined by the relative labour productivities in the jointly produced commodities. This is well known from the analysis of Ricardo's two-country, two-commodity case and it follows immediately from statement (ii) of the Proposition 2. Just fix some arbitrary pair of country indices k_1 and k_2. Let $j \in I_{k_1}(p) \cap I_{k_2}(p)$. Then

$$w_{k_1} a_j^{k_1} = p_j = w_{k_2} a_j^{k_2} \quad \text{which implies} \quad \frac{w_{k_1}}{w_{k_2}} = \frac{1/a_j^{k_1}}{1/a_j^{k_2}}.$$

5. Summary

Jones (1961) characterizes efficient patterns of complete specialization in Ricardian production with many countries and many commodities. His result suffers from not being applicable to the case of incomplete specialization. It has been the object of the present study to close the gap. The set of all efficient patterns of minimal specialization is fully characterized. In particular, it has been shown that Jones' criterion of minimizing products of labour requirements has to be substituted by the criterion of maximizing products of values of labour units to cover the general case. By combining the result of Jones and the present paper one obtains a complete picture of efficient specialization in Ricardian production.

The analysis draws on some recent results of convex geometry and the theory of cephoids. This theory has been developed by Pallaschke and Rosenmüller in a series of papers with the intention to establish a superadditive solution for a well-chosen class of bargaining problems. This class is such that the bargaining sets can be interpreted as the sums of prisms ("cephoids"). Interested readers are referred to Pallaschke and Rosenmüller (2007a, 2007b, 2010) and Rosenmüller (2009).

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