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The Core of an Extended Tree Game: A New Characterisation

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Barbara von Schnurbein[1]

The Core of an Extended Tree Game: A New Characterisation

Abstract

Cost allocation problems on networks can be interpreted as cooperative games on a graph structure. In the classical standard tree game, the cost of a service delivered, by a source has to be allocated between homogeneous users at the vertices. But, modern networks have also the capacity to supply different (levels of) services. For example, a cable network that provides different television standards. Users that choose different levels of service can not be treated equally. The extended tree game accounts for such differences between users. Here, players are characterised by their level of demand, consequently the implications on the cost structure of the problem can be considered. We show how an ET-game can be formulated as the sum of unanimity games. This observation enables us to directly calculate theweighted Shapley values and to identify the core of an ET-game.

JEL Classication: C71, C44

Keywords: Cooperative game theory; extended tree game; core.

October 2010

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1 Introduction.

Networks with heterogeneous users are networks that must be able to satisfy the heterogeneous demand of their users. The heterogeneity results from the fact that users might e.g. demand different levels of quality of the good delivered by the network. The higher then the quality demanded by a user, the higher the costs of connecting this user to the source that provides the service. The costs of establishing or maintaining such a network for different groups of users can be modelled by means of cooperative game theory as an extended tree game (Granot et al., 2002). A solution of an extended tree game then allocates the costs of the network to its users under consideration of their responsibility for the emerged costs (and other equity aspects). These games can be used to model and analyse numerous real networks, whereby the users may differ in their capacity or reliability demand, or might ask for different services. For example, if the network is a pipeline that delivers different quantities of a good in time, then users might ask for different capacities. In the case of a telecommunication network different demands for reliability or even the demand of different services are imaginable. In the theoretical model discussed in this paper the umbrella term *quality demand* shall stand for all possible differences between players.

In extended tree games (ET-games) a tree structure of the network is assumed. The service is delivered by the source and the users are situated at the vertices of the tree. The edges of the tree represent sections of the connection of users to the source. An edge must comply with the quality standard of the user of this edge who has the highest quality demand. This means that the quality demand imposes a quality and a cost structure on the tree network when higher quality of a connection is associated with higher costs. The cost structure is mapped by the cost function, which attributes its costs to each possible union (coalition) of players. The cost allocation problem is concerned with allocating the costs of the complete network between its users. This paper is concerned with the definition of the core of ET-games. Here a different approach to Granot et al. (2002) is chosen, where an algorithm is presented that checks the core membership of an arbitrary allocation. By defining the core as the set of weighted Shapley values, our approach makes it possible to directly name all elements of the core. As such, the given paper is an extension of the work of Granot et al. (2002).

A large selection of literature on cost allocation problems on graph structures is available, some of which are reviewed in Curiel (1997), Sharkey (1995) and Borm et al. (2001). Here, the works concerning standard tree games should be mentioned. Standard tree games (ST-games) consider networks with an established tree structure, where one level of service is disposable from the source. The characteristics and solutions of these games have been discussed e.g. in Bird (1976), Claus and Kleitman (1973), Megiddo (1978) and in Bjørndal et al. (2004). ST-games are relevant for the analysis of ET-games, because ET-games can be additively decomposed in ST-games (Granot et al., 2002). Characteristics of ET-games can be derived from this decomposition, as shown by Granot et al. (2002), who deduce that these games are concave and consequently have a non empty core. In this paper, the additive decomposition will be used in the characterisation of the core of ET-games. As there exists a possibility of formulating the cost function of a ST-game as a weighted sum of the costs of vertices (Koster, 1999), our first challenge will be to develop a similar formula for ET-games. This is the first result of this paper. This formula will lead us to the second result and enable us to directly calculate the weighted Shapley value of the ET-game. As ET-games are concave, every weight system chosen will identify an element of the core of the considered game. Further, the core can be identified by all weighted Shapley values of a game (Monderer et al., 1992). The third and main result of this paper is a new characterisation of the core of ET-games by the set of all weighted Shapley values.

This paper is organised as follows: After the definition of the notation (section 2), we will introduce the ET-games (section 3). There we will not only describe this game and its properties but also describe how the cost function can be formulated here by using unanimity games. This result enables us to directly calculate the Shapley value of an ET-game in section 4. There we also show how the core of an ET-game can be formulated with with weighted Shapley values. Section 5 provides a conclusion.

2 Preliminaries.

A *cost allocation problem* arises if costs of a common project have to be divided amongst its users. Such a problem can be interpreted as a *cost game* (N, c). Where N is the set of users, also called player set $N = \{1, 2, .., n\}$ and $c : 2^N \rightarrow \mathbb{R}$ represents the cost function. A cost function c assigns to each player $i \in N$ (and each coalition of players $S \subseteq N$) the costs of the project satisfying the needs of the considered player $c(\{i\})$ (coalition of players $c(S)$). There are several possible characteristics of cost allocation games. Monotonicity and concavity are relevant for our context. First, a game is monotone if for all $T \subseteq S \subseteq N$:

$$c(T) \leq c(S).$$

In a monotone game the inclusion of new players to a coalition $T \subseteq N$ will never lead to a decrease of the costs of the new coalition S with $T \subseteq S \subseteq N$. Second, a game is concave if for all $i \in N$ and all $T \subseteq S \subseteq N \setminus \{i\}$:

$$c(S \cup \{i\}) - c(S) \leq c(T \cup \{i\}) - c(S).$$

In a concave game the marginal costs of a player i decrease for growing coalitions. It can be shown that ST-games as well as ET-games are monotone and concave.

A solution $\mathbf{x} = (x_1, x_2, ..., x_n)$ of a cost game is the allocation of the costs of the coalition of all players, the grand coalition, $c(N)$ to all players in N. Consequently, the cost shares of all players should sum up to $c(N)$: $\sum_{i \in N} x_i = c(N)$. $\mathbf{x}$ is then called the cost allocation or pre-imputation, and the set of all pre-imputations is denoted by $I^*(N, c)$. An allocation is individually rational if $c(\{i\}) = c_i \geq x_i$ is satisfied for all $i \in N$, then a player never contributes more in the solution $\mathbf{x}$ than his so called stand alone costs c_i. An individually rational allocation is called imputation, and the set of all imputations is denoted by $I(N, c)$. Group rationality requires that all coalitions have to pay less in the solution then the costs of providing the service to its members only, i.e. if for all $S \subseteq N$ holds $\sum_{i \in S} x_i \leq c(S)$. The core $\mathcal{C}(N, c)$ is defined as the set of all pre-imputations that are individually and group rational:

$$\mathcal{C}(N,c) = \{x \in I^*(N,c) | \sum_{i \in S} x_i \leq c(S) \text{ for all } S \subset N\}.$$

Allocations that are elements of the core ensure the stability of the co-operation, as no individual player or coalition of players has an incentive to leave the grand coalition. In the general case, the core of an allocation game can be empty, but the convexity of a cost allocation game assures its non-emptiness.[1]

The core, a set value solution, does not choose a unique solution in the normal case. If a single value solution is desired, it must be chosen out of many alternatives depending of the case under consideration. Such single value solutions are of a special interest if they choose an element of the core, like the Shapley value, a core member in concave games (Shapley, 1971). The Shapley value $\phi = (\phi_1, \phi_2, ..., \phi_n)$ of a cost allocation game can be calculated by the following formula (Shapley, 1953a):

$$\phi_i(c,N) := \sum_{S \subset N} \frac{(|S|-1)!(n-|S|)!}{n!}(c(S) - c(S\backslash\{i\})).$$

Alternatively, the fact that dual unanimity games can serve as a basis for all cooperative cost games can be used to calculate the Shapley value (Koster, 1999).[2] With $S, T \subseteq N$ A dual unanimity game $(N, u_T) \in \mathcal{G}$ is defined by:

Definition 1 (dual unanimity game):

$$u_T^*(S) = \begin{cases} 1, & \text{if} \quad S \cap T \neq \emptyset \\ 0, & \text{else.} \end{cases} \tag{1}$$

Dual unanimity games serve as a basis for the class of all cooperative cost games (N, c), we can write:

$$c = \sum_{T \subseteq N \backslash \{\emptyset\}} \Delta_{T^*} u_T^*. \tag{2}$$

[1] For a proof see e.g. Forgó et al. (1999, p. 323).
[2] For the introduction of unanimity games see also Peleg and Sudhölter (2003, p. 203).

Hereby Δ_{T^*} represents the Harsanyi dividend (Harsanyi, 1963). A Harsanyi dividend Δ_{T^*} of a coalition T can be interpreted as the share of $c(N)$ that can be attributed to the coalition T and has not already been realised by its members by cooperating in smaller coalitions. If a solution distributes each dividend Δ_{T^*} between the players in T according to a given sharing system, it is called a Harsanyi-solution (van den Brink et al., 2007). How this share is distributed to the members of this coalition depends on the solution chosen. The Shapley value of an unanimity game distributes the Harsanyi dividend Δ_{T^*} equally among its $|T|$ members (Shapley, 1953b):

$$\phi_i(N, u_T^*) = \begin{cases} \frac{1}{|T|}, & \text{if} \quad i \in T \\ 0, & \text{else.} \end{cases}$$

The Shapley value of a game (N, v) is derived from the additivity property and from formula 2:

$$\phi_i(N, v) = \sum_{\emptyset \neq S \subseteq N} \Delta_{S^*} \phi_i(N, u_S).$$

A variation of the Shapley value, the weighted Shapley value (or weighted value), distributes the Harsanyi dividends asymmetrically. This solution concept has been designed in order to capture differences between players that are not reflected by the cost function.[3] They are represented by the weight system.

Definition 2 (weight system (Kalai and Samet, 1987)): A Shapley weight system for the game (N, c) is an ordered pair $\mu = (P, w)$. P is an ordered partition of the player set $P = (S_1, ..., S_q)$ and w assigns a weight w_i to player i according to the following definition:

$$w_{S_l} \in int\Delta(S_l), \; \forall \, l = 1, ..., q.$$

$\mathcal{M}(N)$ is the set of all weight systems for the game (N, c).

The players in S_q are interpreted as players with positive weights w_{S_q}. Relative to them the players in $N \setminus S_q$ are players with zero weights. Between

[3] The positively weighted value developed by Shapley (1953b) only allows positive weights. The weighted value introduced by Kalai and Samet (1987) can also consider zero weights.

the players in $N \setminus S_q$, those in S_{q-1} with the weights $w_{S_{q-1}}$ are considered as the heaviest players. They dominate the players in $N \backslash \{S_q \cup S_{q-1}\}$ whenever no player from S_q is present. Generally, the players in S_i $(i = 1, 2, ..., q)$ are the heaviest players in $\cup_{j=1}^{i} S_j$. In order to calculate the relative weights of the players in a coalition S, the heaviest players in this coalition have to be determined. $m(S) = \max\{h | S_h \cap S \neq \emptyset\}$ identifies the index of the heaviest player in S. All heaviest players in S are then in $S \cap S_m =: \overline{S}$. The weighted value of a dual unanimity game u_S^* divides the value of one between the heaviest players in S under consideration of their relative weights:

Definition 3 (dual weighted Shapley value[4]): For a weight system $\mu = (P, w)$ the weighted Shapley value $\phi^\mu : \Gamma \to \mathbb{R}^n$ of an unanimity game is defined by:

$$\phi_i^\mu(N, u_S^*) = \begin{cases} \dfrac{\omega_i}{\omega(\overline{S})}, & \text{if} \quad i \in \overline{S} \\ 0, & \text{else.} \end{cases}$$

Derived from the additivity of the Shapley value, the value of a game (N, c) is:

$$\phi_i^\mu(N, c) = \sum_{\emptyset \neq S \subseteq N} \Delta_S \phi_i^\mu(N, u_S^*).$$

3 Extended Tree Games.

3.1 Description and Properties.

In the following section 3 we will briefly describe ET-games and their properties as presented by Granot et al. (2002). Then, in section 3.2. we will develop a new representation of ET-games in which unanimity games will be used. Let $G = (E, V)$ be a tree graph. The set of vertices $V = \{N \cup \{r\}\}$ consists of the player set N and the root r. $E = \{e_1, e_2, .., e_n\}$ represents the set of edges, where e_i is the unique edge emanating from i and on the unique path from i to r in G. Each player is of a type $p \leq n$. The type of a player i is denoted by γ_i. The type of a player i defines his requirement on the quality of his connection to the root. A player i of type $\gamma_i = l$ requires that all edges on his unique path to the root are of type l (or higher). Further, the costs of a connection depend on the quality for which it is designed for. The construction of an edge e_i for the type l costs a_i^l. The costs a_i^γ are assumed to be monotone in types, i.e. for all $i \in N$ $0 \leq a_i^1 \leq a_i^2 \leq ... \leq a_i^p$. If a player is further of type $\gamma_i = 0$, connecting him to the root is assumed to be free $a_i^0 := 0$.

For each $i \in N$, $F(i)$ denotes the set of players on the subtree G_i of G that is routed in i. We call the elements of $F(i)$ followers of player i. These are players situated on the vertices that follow i on the unique way to the root.[5] For each coalition $S \subseteq N$ the maximal quality requirement in S is given by $\gamma(S) := \max\{\gamma_i | i \in S\}$. Consequently, the quality of the connection e_i in an optimal tree for $S \subseteq N$ is defined by $\gamma(S \cap F(i))$ in order to satisfy the quality requirements of the members of coalition S. In an optimal graph, the quality of an edge must be exactly as high as the highest quality requirement of all players using that edge. If all players are of type 1, the ET-game reduces to a ST-game (Megiddo, 1978).

Granot et al. (2002) show that an ET game can be formulated as a sum of simple ET-games with players of type 1 and 0. For each $\gamma = l$ let N^γ be the set of all players of type γ. For $\gamma = 1,..,p$ (N, c^γ) denotes an ET game by designating $N^0 \cup N^1 \cup ... \cup N^{\gamma-1}$ as 0-players and $N^\gamma \cup ... \cup N^p$ as 1-players with the costs of the edge e_i given by the marginal costs of the quality amelioration from $\gamma - 1$ to γ: $g_i^\gamma = a_i^\gamma - a_i^{\gamma-1}$ (*marginal costs of quality*). In an ET-game (N, c^l) the players of type 1 are denoted by $N^{*l} = \bigcup_{\gamma=l}^p N^\gamma$. For the ET-game (N, c), $c = \sum_{\gamma=1}^p c^\gamma$.

It can also be shown that such simple ET-games correspond to ST-games (Granot et al., 2002). Let (N, c) be an ET-game with 0-players in the set N^0 and 1-players in the set N^1. If a reduced game $(N^1, \hat{c})$ is defined in which only the 1-player are considered then it can be seen that $\hat{c}(S) = c(S)$ for all $S \subseteq N^1$, because 0-players are dummies and never generate costs.

[5] In the next chapter the assumption that each vertex is occupied by exactly one player will be relaxed. It will be allowed that a vertex is occupied by one or no player. N_v describes then the set of players situated on the vertex v, and the N_{V_1} describes the set of players situated at the vertices in the set V_1. $F(v)$ denotes analogically the set of vertices following v.

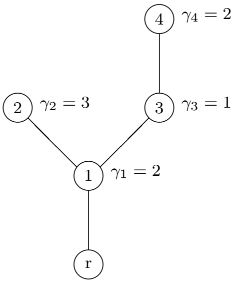

Fig. 1: Extended Tree with 4 players.

The restricted game $(N^1, \hat{c})$ is a ST-game. Since ST-games are monotone and concave, it follows that a reduced game with the two types of players is also monotone and concave. Further, since the sum of monotone and concave games is also monotone and concave, it follows that every ET-game is monotone and concave. From Shapley (1971) it follows that the core of an ET-game is not empty and that the Shapley value is contained in the core.

Example 1: For demonstration of the decomposition of an ET-game in ST-games consider the example in Figure 1. In a tree graph with a root r, 4 players are situated at the vertices. Their quality requirements are $\gamma_1 = 2$, $\gamma_2 = 3$, $\gamma_3 = 1$ and $\gamma_4 = 2$.

With 3 quality levels, the game in Figure 1 can be decomposed into three subgames. Each subgame corresponds to one quality level. We call the 1-players or relevant players in the subgame corresponding to γ $N^{*\gamma}$: for $\gamma = 3$ we have $N^{*3} = \{2\}$, for $\gamma = 2$ $N^{*2} = \{1, 2, 4\}$ and $N^{*1} = \{1, 2, 3, 4\}$ for $\gamma = 1$. The player set N in each subgame in Figure 2 is identical, the costs on the edges differ according to the definition of marginal costs of quality. We refer to these games as (N, c^1),(N, c^2) and (N, c^3). The game (N, c^1) is a ST-game as there are only 1-players. In the game (N, c^2) there is one 0-player, the player 3. If he is excluded from the player set, we can write the new cost function $\hat{c}^2(S) = c^2(S)$ for all $S \subseteq N^{*2}$. The game $(N^{*2}, \hat{c}^2)$ is then a ST-game. The ET-game (N, c^3) can be reduced in a similar way. The ST-game $(N^{*3}, \hat{c}^3)$ has only one player and $\hat{c}^3(S) = c^3(S)$ for all $S \subseteq N^{*3}$, i.e. $c^3(\{2\}) = \hat{c}^3(\{2\})$.

In the next section our first result will be developed. We will show how the

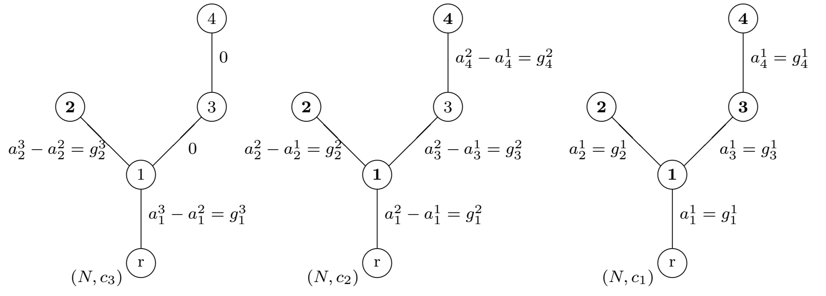

Fig. 2: Decomposition of the ET in Figure 1: player of type **1 boldface**; player of type 0 normal type.

cost functions of ET-games can be denoted with the help of unanimity games due to the additive decomposition.

3.2 Sum of Unanimity Games.

In this section we will show how an ET-game can be formulated as a weighted sum of dual unanimity games with the marginal costs of arcs as weights. In order to demonstrate this we have to relax the assumption that each vertex is populated by exactly one player. We assume that at any one vertex there can be situated one or no player, as described in chapter 3.1.

First, we have to consider ST-games. These games have an intuitive formulation using unanimity games as shown by Koster (1999). The cost function of a ST-game is described by:

$$c(S) = \sum_{v \in T_S} c(e_v),\ \forall\, \emptyset \neq S \subseteq N,$$

where $T_S = \{v \in V | \exists v' \in V, N_{v'} \cap S \neq \emptyset \text{ und } v \preceq v'\}$ denotes the so called trunk of the coalition S. This definition can be transformed to:

$$c(S) = \sum_{v \in \{V | S \cap N_{F(v)} \neq \emptyset\}} c(e_i),\ \forall\, \emptyset \neq S \subseteq N, \tag{3}$$

as $N_{F(v)}$ is the set of players situated at the vertices following v. From definition 2.1 we know that $u^*_{N_{F(v)}}(S)$ is equal to 1 only if $S \cap N_{F(v)} \neq 0$.

Hence the ST-game $(N, c_{\mathcal{G}})$ can be represented as

$$c = \sum_{v \in V \setminus \{r\}} c(e_v) u^*_{N_{F(v)}}. \tag{4}$$

This representation allows for the description of ST-games with non populated vertices as the costs of edges are only attributed to players that use them.[6]

With these definitions, we can face the challenge of formulating ET-games as a sum of unanimity games. In order to achieve this, we have to focus on the subgames for each quality level $\gamma = 1, .., d$. N^{*l} describes the set of 1-players in the subgame $\gamma = l$. They are also called relevant players. In each subgame the set of relevant followers of a vertex v is denoted by:

$$F^{*l}(v) = F(v) \cap N^{*l}.$$

Choosing the relevant followers as the basis of the unanimity game in the representation of a subgame allows the consideration of all vertices, which is useful in representing the complete ET-game. We can write for a subgame for $\gamma = l$:

$$c^l = \sum_{v \in V \setminus \{r\}} g^l_v(e_v) u^*_{N_{F^{*l}(v)}}. \tag{5}$$

This representation allows us to define the cost function of an ET-game by summing all subgames:

Definition 4: The cost function of an ET-game with the players set N and the quality demands $\gamma = 1, ..., d$ of the players is given by:

$$c^{ET} = \sum_{\gamma=1}^{d} c^\gamma = \sum_{\gamma=1}^{d} \sum_{v \in V \setminus \{r\}} g^\gamma_v(e_v) u^*_{N_{F^{*\gamma}(v)}}. \tag{6}$$

The following example illustrates this definition.

Example 2: In the tree graph in Figure 3 the players have the following quality demands: player 1: $\gamma_1 = 1$, player 2: $\gamma_2 = 3$, player 3: $\gamma_3 = 2$.

[6] For an example see Koster (1999, p. 172).

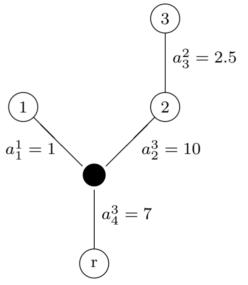

Fig. 3: Extended Tree in example 2.

The costs of each edge, given all quality levels, are summarized in Table 1, where the marginal costs of quality g_v^l are also denoted. The vertices numbered from 1 to 4. Their labels correspond to the labels of the players situated at these vertices, the non populated vertex is referred to as 4.

Kante:	$\gamma = 1$	$\gamma = 2$	$\gamma = 3$	g_v^3	g_v^2	g_v^1
v=1	1	2	5	3	1	1
v=2	3	5	10	5	2	3
v=3	2	2,5	4	1,5	0,5	2
v=4	1,5	3	7	4	1,5	1,5

Tab. 1: Example 2: costs of the edges a_v^γ

The quality demand defines the optimal network for the grand coalition as summarised in Figure 3. The decomposition in subgames is demonstrated in the Figure 4. So, three subgames result: c^3, c^2 and c^1. Each subgame is defined on the player set N, 0-players are treated as dummies or *non relevant* players. N^{*l} denotes the set of *relevant players* for each γ. For $\gamma = 3$ there is one relevant player $N^{*3} = \{2\}$, for $\gamma = 2$ two players are relevant $N^{*2} = \{2, 3\}$ and for the third subgame for $\gamma = 1$ all players in $N^{*1} = N = \{1, 2, 3\}$ are relevant.

The cost function of the subgames can be formulated as:

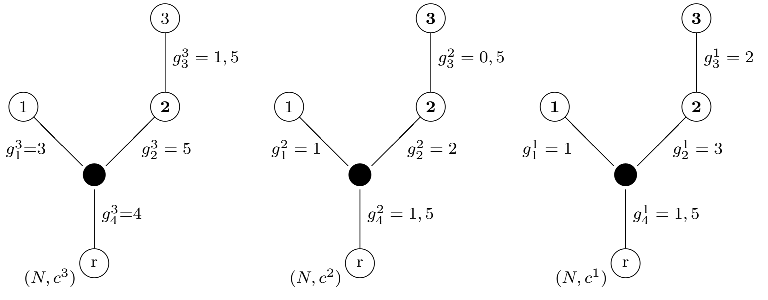

Fig. 4: Decomposition of the ET in example 2: player of type **1 boldface**; player of type 0 normal type.

subgame $\gamma = 3$

$$\begin{aligned} c^3 &= \sum_{v \in V \setminus \{r\}} g_v^3 u^*_{N_{F^{*3}(v)}} \\ &= 3u^*_\emptyset + 1,5u^*_\emptyset + 5u^*_{\{2\}} + 4u^*_{\{2\}}, \end{aligned} \tag{7}$$

subgame $\gamma = 2$

$$\begin{aligned} c^2 &= \sum_{v \in V \setminus \{r\}} g_v^2 u^*_{N_{F^{*2}(v)}} \\ &= 1u^*_\emptyset + 0,5u^*_{\{3\}} + 2u^*_{\{2,3\}} + 1,5u^*_{\{2,3\}}, \end{aligned} \tag{8}$$

subgame $\gamma = 1$

$$\begin{aligned} c^1 &= \sum_{v \in V \setminus \{r\}} g_v^1 u^*_{N_{F^{*1}(v)}} = \sum_{v \in V \setminus \{r\}} a_v^1 u^*_{N_{F(v)}} \\ &= 1u^*_{\{1\}} + 2u^*_{\{3\}} + 3u^*_{\{2,3\}} + 1.5u^*_{\{1,2,3\}}. \end{aligned} \tag{9}$$

For demonstration, we show how the value of coalition $\{1, 3\}$ is calculated in each subgame. In the subgame $\gamma = 3$ both players are non relevant. The cost function c^3 should not attribute any costs to them. As $u^*_\emptyset = 0$ we know from formula 7:

$$c^3(\{1, 3\}) = 3 \cdot 0 + 1,5 \cdot 0 + 5 \cdot 0 + 4 \cdot 0 = 0.$$

In the subgame $\gamma = 2$ player 1 is non relevant, and player 3 is relevant. The costs of the coalition $\{1, 3\}$ in this subgame are:

$$c^2(\{1,3\}) = 1 \cdot 0 + 0,5 \cdot 1 + 2 \cdot 1 + 1,5 \cdot 1 = 4.$$

In this case, $c^2(\{1,3\}) = c^2(\{3\})$. In the subgame $\gamma = 1$ both players are relevant:

$$c^1(\{1,3\}) = 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 + 1,5 \cdot 1 = 7,5.$$

In order to calculate the costs of this coalition in the complete ET-game the values for the subgame can be summed up:

$$c^{ET}(\{1,3\}) = \sum_{\gamma=1}^{3} \sum_{v \in V \setminus \{r\}} g_v^{\gamma}(e_v) u^*_{N_{F^{*\gamma}}} = 0 + 4 + 7,5 = 11,5.$$

Using unanimity games to formulate ET-games is our first result. This allows us to identify the core of these games in the next section.

4 The Core of an Extended Tree Game.

Monderer and Samet (2002) show that in convex games the set of all weighted Shapley-values defines the core. Bjørndal et al. (2004, p. 264) give a constructive proof of this result for special concave cost games, the ST-games. We will use this result and our result of the previous section to identify the core of ET-games. First, we will develop a definition of the weighted Shapley value of ET-games, which is the second result of this paper. Therefore, the set of heaviest players has to be defined for all subgames under consideration of only the relevant players.

Definition 5: In an ET-game on the graph G with the player set N, the set of vertices V and quality demands of the players $\gamma = 1, ..., p$ the relevant heaviest players in a coalition $S \subseteq N$ are given by:

$$S^{*\gamma}(v) := N_{F^{*\gamma}(v)} \cap S_{\max\{j \mid N_{F^{*\gamma}(v)} \cap S_j \neq \emptyset\}}.$$

The weighted value of a subgame γ is defined by the values of the unanimity games for each edge and its followers:

$$\phi_i^{\omega}(N, u^*_{N_{F^{*\gamma}(v)}}) = \begin{cases} \dfrac{\omega_i}{\omega(S^{*\gamma}(v))}, & \text{for} \quad i \in S^{*\gamma}(v) \\ 0, & \text{else.} \end{cases}$$

For player i in a subgame γ the weighted Shapley value is given by :

$$\phi_i^{\omega}(N, c^{\gamma}) = \sum_{v \in V \setminus \{r\}} g_v^{\gamma}(e_v) \phi_i^{\omega}(N, u^*_{N_{F^{*\gamma}(v)}}), \qquad (10)$$

because of additivity and formula 5.

In an ET-game with $\gamma = 1, ..., p$ quality levels the additive decomposition of ET-games and formula 6 lead us to the following definition of the Shapley value of player i:

Definition 6: The weighted Shapley value of player i in an ET game is the sum of the weighted Shapley values in the subgames $\gamma = 1, ..., p$:

$$\phi_i^{\omega}(N, c^{ET}) = \sum_{\gamma=1}^{p} \sum_{v \in V \setminus \{r\}} g_v^{\gamma}(e_v) \phi_i^{\omega}(N, u^*_{N_{F^{*\gamma}(v)}}). \qquad (11)$$

In order to calculate the weighted value of an ET-game, the subgames have to be considered first. While the set of followers of each edge does not change for different subgames, the other characteristics usually do. So, the set of relevant players N^{*l} and for each edge, the set of relevant senior players $S^{*l}(v)$ as well as the marginal costs of quality g_v^l, are specific for the subgame $\gamma = l$. In the following example 3 the graph in Figure 4 will be considered in order to demonstrate the calculation of a weighted Shapley value.

Example 3: We consider the tree graph in Figure 4 and calculate the weighted Shapley vector for the weight system $\mu_1 = ((\{1, 3\}, \{2\}), (0,75; 1; 0,25))$. The relevant elements of the description of the resulting subgames as well as the necessary steps of calculation are specified in Table 2. The upper three Tables describe the three subgames: the subgame for $\gamma = 3$ in the first, for $\gamma = 2$ in the second and the $\gamma = 3$ in the third Table. These three Tables are constructed analogically. For each vertex (column 1) the set of followers (column 2), the set of heaviest relevant players (column 3) and the marginal costs of the considered quality (column 4) are given. In column

5-7 the weighted Shapley values are calculated for the unanimity games with the basis $F^{*\gamma}(v)$ corresponding to each vertex v. Further, in the lowest row the Shapley values of the subgame are computed by weighting and summing up the values of the respective unanimity games. The lowest Table sums up the values for the subgames to the weighted Shapley values of the considered ET-game.

In the given weight system player 2 dominates the remaining players. The chosen partition does not affect the calculation in the subagme $\gamma = 3$ because player 2 is the only relevant player. The costs of 9 are consequently entirely attributed to him. In subgame $\gamma = 2$ there are two relevant players 2 and 3. Player 2 as the senior player according to the weight system, bears the costs of each connection on his path to the source. Only the costs of e_3, an edge, that is only used by player 3, are not attributed to him. In the last subgame $\gamma = 1$ all players are relevant. Here the senior player 2 bears the costs of his connection to the source as well. The costs of the remaining edges used either only by player 1 or by player 3 are respectively assigned to them. The Shapley vector of the ET-game results as the sum of the vectors of the subgames $\phi_i^{\omega}(N, c^{ET}) = (1; 17; 2, 5)$. Here the senior player is charged with his stand alone costs while the other players only have to pay for the connections constructed exclusively for them.

It has been shown in example 3 how the weighted Shapley value can be calculated in ET-games according to formula 2. Also as, already pointed out, a weighted Shapley value will always be an element of the core. But our approach does not only allow the identification of single core elements by choosing alternative weight systems. Instead, all elements of the core can be described with this approach. As we discussed already in concave games the set of weighted Shapley values equals the core.

Proposition 1: The core C of an ET-game $(N, c_{\mathcal{G}})$ is equal to the set of the weighted Shapley values: $\{\phi^{\mu}(c_{\mathcal{G}}^{ET}) | \mu \in \mathcal{M}(N)\} = \mathcal{C}(c_{\mathcal{G}})$.

5 Conclusions.

In the present paper we have discussed ET-games (Granot et al., 2002). We have shown, how additive decomposition can be used in the formal description of these games. This formal description allows a direct calculation of the weighted Shapley values. And since the set of all Shapley values defines the core of an ET-game, it also provides the definition of the core.

v	$N_{F(v)}$	S^{*3}	g_v^3	$\phi_1^\mu(N, u^*_{N_{F^{*\gamma}(v)}})$	$\phi_2^\mu(N, u^*_{N_{F^{*\gamma}(v)}})$	$\phi_3^\mu(N, u^*_{N_{F^{*\gamma}(v)}})$
1	$\{1\}$	$\emptyset$	3	0	0	0
2	$\{2,3\}$	$\{2\}$	5	0	1	0
3	$\{3\}$	$\emptyset$	1,5	0	0	0
4	$\{1,2,3\}$	$\{2\}$	4	0	1	0
			$\phi^\mu(N, c^3_{\mathcal{G}})$	0	9	0

subgame $\gamma = 3$; $N^{*3} = \{2\}$.

v	$N_{F(v)}$	S^{*2}	g_v^2	$\phi_1^\mu(N, u^*_{N_{F^{*\gamma}(v)}})$	$\phi_2^\mu(N, u^*_{N_{F^{*\gamma}(v)}})$	$\phi_3^\mu(N, u^*_{N_{F^{*\gamma}(v)}})$
1	$\{1\}$	$\emptyset$	1	0	0	0
2	$\{2,3\}$	$\{2\}$	2	0	1	0
3	$\{3\}$	$\{3\}$	0,5	0	0	1
4	$\{1,2,3\}$	$\{2\}$	1,5	0	1	0
			$\phi^\mu(N, c^2_{\mathcal{G}})$	0	3,5	0,5

subgame $\gamma = 2$; $N^{*2} = \{2,3\}$.

v	$N_{F(v)}$	S^{*1}	g_v^1	$\phi_1^\mu(N, u^*_{N_{F^{*\gamma}(v)}})$	$\phi_2^\mu(N, u^*_{N_{F^{*\gamma}(v)}})$	$\phi_3^\mu(N, u^*_{N_{F^{*\gamma}(v)}})$
1	$\{1\}$	$\{1\}$	1	1	0	0
2	$\{2,3\}$	$\{2\}$	3	0	1	0
3	$\{3\}$	$\{3\}$	2	0	0	1
4	$\{1,2,3\}$	$\{2\}$	1,5	0	1	0
			$\phi^\mu(N, c^1_{\mathcal{G}})$	1	4,5	2

subgame $\gamma = 1$; $N^{*1} = N$.

Spieler	$\phi_i^\mu(N, c^3_{\mathcal{G}})$	$\phi_i^\mu(N, c^2_{\mathcal{G}})$	$\phi_i^\mu(N, c^1_{\mathcal{G}})$	$\phi_i^\mu(N, c^{ET})$
1	0	0	1	1
2	9	3,5	4,5	17
3	0	0,5	2	2,5

Tab. 2: Calculation of the weighted Shapley value for the graph in Figure 4 with $\mu_1 = ((\{1,3\},\{2\}),(0,75;1;0,25))$.

Hence, our approach adds new aspects to the analysis of ET-games developed by Granot et al. (2002). These authors have shown how the core membership of an arbitrary allocation can be checked. Our approach allows us to calculate arbitrary elements of the core by choosing different weight systems as well as a new mapping of the core of ET-games. Here, the core is not described by a system of equations (Granot et al., 2002) but by the set of all its elements.

Direct calculation of weighted Shapley values of (the concave) ET-games allows the identification of a single value solution that is an element of the core. Even though the choice of the weight system makes it possible to consider arbitrary asymmetries between the players, incentive compatibility will always be guaranteed by a weighted Shapley value in ET-games.

The ET-games offer a theoretically interesting set of tools for the analysis of networks with heterogenous users. Areas for further research are the empirical applications of the developed theory. These are closely connected to the question of how such networks with very numerous users can be modeled, and how real networks can be approached by a tree structure. An interesting application would be to the calculation of energy transmission costs on the international (e.g. European) level. Here, physical energy flows can be very heterogeneous, and it is reasonable to consider this in the cost allocation system. The fact that weighted Shapley values are elements of a core in the discussed games could facilitate the finding of a political compromise on the multinational level.

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