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Abstract

This paper argues that cross-sectional dependence (CSD) is an indicator of misspecification in panel quantile regression (QR) rather than just a nuisance that may be accounted for with panel-robust standard errors. This motivates the development of a novel test for panel QR misspecification based on detecting CSD. The test possesses a standard normal limiting distribution under joint N, T asymptotics with restrictions on the relative rate at which N and T go to infinity. A finite-sample correction improves the applicability of the test for panels with larger N. An empirical application to housing markets illustrates the use of the proposed cross-sectional dependence test.

JEL-Codes: C12, C23

Keywords: Cross-unit correlation; conditional quantile; factor model; exogeneity

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1 Introduction

Compared to cross-sectional data, panel data analyses offer the opportunity to deal with data issues such as unobserved heterogeneity. Similarly, typical difficulties arising in time series contexts, say short samples and instabilities, may also be sidestepped in a panel setup. Panel data however prompt specific challenges, of which cross-sectional error dependence is among the more important ones. Cross-sectional dependence may arise for various reasons, most prominently due to global shocks affecting several units at the same time. The dramatic effects on the asymptotic and finite-sample properties of the least-squares [LS] estimator and standard inferential procedures have been discussed in the literature; see e.g. Andrews (2005). In particular, should the regressors correlate with the global shocks, endogeneity is expected to bias the LS estimator. Even if endogeneity is not an issue,[1] the variances of the panel estimators are typically affected by the presence of cross-sectional dependence. Therefore, detecting and accounting for cross-dependence is a necessary step in panel data analyses. This step is by no means a secondary one; see, for instance, Pesaran (2004), the survey of Chudik and Pesaran (2015) or Bai et al. (2016) and references therein.

A strand of panel literature gaining momentum is dedicated to panel quantile regressions [QR]; see, for instance, Koenker (2005, Section 8.7) or Chernozhukov et al. (2013). For early applications of quantile panel data methods, see, among others, Abrevaya and Dahl (2008); Kniesner et al. (2010); Gamper-Rabindran et al. (2010); Covas et al. (2014); Binder and Coad (2015). More recently, Zhu et al. (2016) use panel QR to analyse the impact of foreign direct investment (FDI), economic growth and energy consumption on carbon emissions in five selected member countries in the Association of South East Asian Nations; Martínez-Zarzoso et al. (2017) investigate whether aid for trade leads to greater exports in recipient countries; Opoku and Aluko (2021) use it to analyse the heterogeneous effect of industrialization on the environment; Baruník and Čech (2021) investigate how to measure common risks in the tails of return distributions using panel QR, while Brownlees and Souza (2021) and Nandi (2022) take a panel route to multi-country Growth-at-Risk.

[1]See Kapetanios, Serlenga, and Shin (Kapetanios et al.) for a recent test of factor exogeneity.

On the theory side, the asymptotic analysis provided by Kato et al. (2012) emphasizes the role of the relation between the time and cross-sectional dimensions of the panel. Harding and Lamarche (2014) allow for a factor structure in the disturbances (see also Pesaran, 2006 and Bai, 2009) where factors, loadings and regressors are not independent, and propose a suitable IV estimator (see also Harding et al., 2020). Still, in spite of the increased use and development of QR methods, the effect of cross-sectional dependence in panel QR has to date not been fully explored yet.

This paper's contribution to the literature is two-fold. First, we argue that cross-sectional dependence is far less benign in QR than in LS regressions. Concretely, we show that a factor structure in the errors may induce asymptotic bias in the panel QR slope parameter estimators even if the factors and loadings are independent of all other model components – unlike LS under the same circumstances. The explanation for this perhaps counter-intuitive finding is that the omitted factors shift the conditional quantile of the idiosyncratic errors in a way that does depend on the regressors in general, and thus have an *indirect* confounding effect on the panel QR estimator. In the LS regression framework, only the standard errors are affected under such exogeneity scenarios, and panel-robust standard errors (Arellano, 1987; Driscoll and Kraay, 1998) are widely used in practice to deal with cross-correlation. However, QR counterparts of clustered standard errors (see Parente and Santos Silva, 2016; Yoon and Galvao, 2016) only account for cross-sectional error dependence if cross-dependence does not induce asymptotic biases in the slope coefficient's estimators.

Second, we discuss ways of testing the null hypothesis of no cross-sectional error dependence in panel QR models. Apart from their original use as detectors of cross-sectional dependence (say in order to decide on whether to use the usual or panel-robust standard errors), such procedures also play the important role of misspecification tests in panel QR. In LS regression models, a factor structure of the errors only causes endogeneity bias if the factors correlate with the regressors. Since, as we show here, biases may arise in panel QR irrespective of any dependence between common error components and regressors, any form of cross-dependence is therefore indicative of misspecification. A cross-dependence test is not a replacement for standard specification procedures such as Hausman tests.

The latter are however more demanding, requiring the existence of exogenous instruments which may be costly to obtain. Therefore, detecting cross-sectional dependence is a reasonable and convenient model check, and, in this sense, we provide a procedure which complements standard specification tests. It should of course be emphasized that, when cross-sectional dependence is found, one should resort to estimation methods accounting for its presence; see e.g. Harding and Lamarche (2014) and Chen et al. (2021).

We proceed as follows. In Section 2, we illustrate the biasing effect of ignoring cross-sectional dependence on fixed-effects panel QR; the effect appears even if factors and loadings are strictly exogenous, which is in stark contrast to the LS case. Moreover, the arguments extend to nonlinear GMM panel procedures, indicating that panel LS estimation is rather the particular case where cross-dependence is benign under exogeneity of the common error components. We then discuss in section 3 the adaptation of the residual-based Breusch-Pagan test (Breusch and Pagan, 1980) of no cross-sectional dependence to the QR framework of this paper, provide joint N, T asymptotics and propose a finite-sample correction. The proposed cross-sectional dependence tests are valid for panel QR estimators satisfying weak regularity conditions. Section 4 analyzes the finite-sample properties of the new tests, and we illustrate our procedures in an application to housing markets in Section 5. The final section concludes, and technical proofs of the results stated throughout the paper are provided in an appendix, together with additional empirical findings.

2 Effects of cross-sectional dependence

We are interested in the τth conditional quantile of $y_{i,t}$ and consider the "structural" model

$$y_{i,t} = \alpha_{i,\tau} + \boldsymbol{\beta}'_\tau \boldsymbol{x}_{i,t} + u_{i,t,\tau} \tag{1}$$

where the subscript τ on the model parameters indicates that coefficients may change across quantiles. The disturbances $u_{i,t,\tau}$ have a factor structure such that,

$$u_{i,t,\tau} = \boldsymbol{\lambda}'_{i,\tau} \boldsymbol{f}_t + \varepsilon_{i,t,\tau}. \tag{2}$$

Such common components may arise e.g. due to global shocks or even omitted variables. The idiosyncratic errors $\varepsilon_{i,t,\tau}$ have zero τ-quantile conditionally on $\boldsymbol{x}_{j,s}$, $\forall\ j = 1, \ldots, N$ and $s = 1, \ldots, T$. Factor models of this type have been recently discussed by Chen et al. (2021); see also Tran et al. (2019) for a less parametric approach.

Irrespective of the concrete estimation method used, the asymptotic properties of the estimators $\hat{\boldsymbol{\beta}}_\tau$ of the slope coefficients in (1) rely on a correct model specification in which the "aggregate" errors $u_{i,t,\tau}$ have zero conditional τth quantile given the regressors $\boldsymbol{x}_{i,t}$. This is, however, not guaranteed to occur in error models of the kind formulated in (2), even if the unobserved variables $\boldsymbol{f}_t$ are strictly exogenous.

To illustrate the fact that cross-dependence, as induced by the latent component $\boldsymbol{f}_t$, may have unexpected effects in the panel QR in (1), let us focus on the simplest model with one regressor and a scalar factor, whose impact, for simplicity, does not depend on the quantile, $\boldsymbol{\lambda}_{i,\tau} = \lambda_i$, i.e.,

$$y_{i,t} = \alpha_{i,\tau} + \beta_\tau x_{i,t} + \lambda_i f_t + \varepsilon_{i,t,\tau}.$$

Furthermore, let $\{f_t\}$ be independent of $\{\varepsilon_{i,t,\tau}\}$, $\{x_{i,t}\}$ and the fixed effects $\{\alpha_{i,\tau}\}$. Just to make the point, take $\varepsilon_{i,t,\tau}$ to be normal (conditionally on the regressors x) with mean $m_{i,t}$ and variance $\sigma^2_{i,t}$, and let f_t be normal with mean m and variance σ^2. Note that it must hold that

$$m_{i,t} + z_\tau \sigma_{i,t} = 0$$

for the conditional τ-quantile of $\varepsilon_{i,t,\tau}$ to be zero, where z_τ is the τ-quantile of the standard normal distribution. Clearly, the setup of this illustration is quite specific, if not oversimplifying. At the same time it pinpoints the impact of cross-sectional dependence in panel QR regressions with minimal technical effort.

Under these conditions, $u_{i,t,\tau}$ is (conditionally) normal as well. Denote the corresponding conditional τ-quantile by $q_{i,t,\tau}$, which obtains as

$$q_{i,t,\tau} = m_{i,t} + m\lambda_i + z_\tau\sqrt{\sigma_{i,t}^2 + \lambda_i^2\sigma^2}.$$

There is no omitted variable bias whenever this conditional quantile does not depend on the regressor x. However, it holds that

$$\begin{aligned} q_{i,t,\tau} &= m_{i,t} + z_\tau\sigma_{i,t} + m\lambda_i + z_\tau\left(\sqrt{\sigma_{i,t}^2 + \lambda_i^2\sigma^2} - \sigma_{i,t}\right) \\ &= m\lambda_i + z_\tau\left(\sqrt{\sigma_{i,t}^2 + \lambda_i^2\sigma^2} - \sigma_{i,t}\right), \end{aligned}$$

where we used the fact that $m_{i,t} + z_\tau\sigma_{i,t} = 0$. The first component, $m\lambda_i$, is absorbed into the fixed effect $\alpha_{i,\tau}$ as long as m does not depend on x (which we excluded to make the point). Should the second component of $q_{i,t,\tau}$ also not depend on t, there is no omitted variable bias, at least not in the slope coefficient estimators (the fixed effects are treated here as nuisance parameters and any bias in the fixed effects estimators may thus be ignored). Moreover, there is no bias in the slope coefficients whenever $z_\tau = 0$, i.e. for median regressions in this example.

But, apart from the case $z_\tau = 0$, one may expect effects on the conditional quantile of the $u_{i,t,\tau}$, when the $\varepsilon_{i,t,\tau}$ are systematically heteroskedastic. If conditional heteroskedasticity is present, say $\sigma_{i,t}^2 = \sigma_{i,t}^2(x_{i,t})$, the conditional quantiles of the errors $u_{i,t,\tau}$,

$$q_{i,t,\tau} = m\lambda_i + z_\tau\left(\sqrt{\sigma_{i,t}^2(x_{i,t}) + \lambda_i^2\sigma^2} - \sigma_{i,t}(x_{i,t})\right),$$

depend explicitly on $x_{i,t}$, and the linear QR model $y_{i,t} = \alpha_{i,\tau} + \beta_\tau' x_{i,t} + error$ is misspecified.

Effectively, one is dealing with an artificially induced nonlinear functional form, since the data generating process is,

$$\mathrm{P}\left(y_{i,t} \leq c_i + \beta_\tau x_{i,t} + z_\tau\left(\sqrt{\sigma_{i,t}^2(x_{i,t}) + \lambda_i^2\sigma^2} - \sigma_{i,t}(x_{i,t})\right)\right) = \tau.$$

At the same time, (1) specifies a linear model to be fitted, resulting in misspecification bias.

The resulting bias of the slope parameter estimators depends on the strength of the cross-sectional dependence (as captured by the nonzero λ_i) and on the marginal distribution of the regressors. Moreover, its magnitude is expected to be larger for more extreme quantiles.

Remark 1. Such effects have been noticed before in a more restricted context: for instance, quantile fixed effects regressions and quantile random effects regressions do not estimate the same quantity (see e.g. the discussion in Galvao and Poirier, 2019). In a similar vein, Hausman et al. (2021) discuss the estimation of QR models with measurement errors in the dependent variable. Ultimately, the issue boils down to the quantile not being a linear operator, unlike the expectation. $\diamondsuit$

Remark 2. One may obtain more concrete statements on the misspecification bias if considering "small" loadings λ_i. Concretely, as $\lambda_i \to 0$,

$$z_\tau \left(\sqrt{\sigma_{i,t}^2 + \lambda_i^2 \sigma^2} - \sigma_{i,t} \right) = z_\tau \frac{\lambda_i^2 \sigma^2}{2\sigma_{i,t}\left(x_{i,t}\right)} + o\left(\lambda_i^2\right),$$

so, assuming e.g. that $\sigma_{i,t}\left(x_{i,t}\right) = \gamma / x_{i,t}$ with $x_{i,t} > 0$ a.s. and $\lambda_i = \lambda$, we obtain errors $u_{i,t,\tau}$ having conditional quantile

$$q_{i,t,\tau} = m\lambda_i + z_\tau \frac{\lambda^2 \sigma^2}{2\gamma} x_{i,t} + o\left(\lambda^2\right),$$

which, under regularity conditions ensuring $\sqrt{NT}$-consistency of $\hat{\boldsymbol{\beta}}_\tau$, suggests that

$$\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau - z_\tau \frac{\lambda^2 \sigma^2}{2\gamma} = o\left(\lambda^2\right) + O_p\left(\frac{1}{\sqrt{NT}}\right).$$

The conclusion (with a different expression for the bias) arguably holds for more general forms of heteroskedasticity and also for non-normal errors. For instance, should $\sigma_{i,t}$ be a function of time rather than depend on $x_{i,t}$, cross-sectional dependence would induce a time trend at the τth quantile. Furthermore, we note that already a magnitude order of $N^{-1/4}T^{-1/4}$ for the loadings λ_i may lead to such (2nd-order) biases in the Gaussian case. $\diamondsuit$

Remark 3. The same line of argumentation indicates that GMM panel estimators based on moment conditions that are nonlinear in the errors are affected by cross-sectional dependence in a similar manner. Finally, the effect of ignored dependence is expected to be similar for nonlinear panel QR models, even if an exact quantification is more difficult than in the presented linear panel QR example.

$\diamond$

It may be seen that the biasing effect of ignored cross-dependence is not specific to pooled estimation, since the shift in the conditional error quantile would equally affect individual-unit estimation, and in fact in a unit-specific way depending on the loadings λ_i. Relatedly, we also note that ignored slope coefficient heterogeneity may induce cross-dependence too, e.g. when regressors are cross-dependent themselves.

Summing up, detecting cross-sectional dependence in panel QR is of paramount importance in applied work. The following section discusses a test of no cross-sectional dependence for specific use with panel QR.

3 Tests of cross-sectional dependence in panel QR

Should one observe the disturbances $u_{i,t,\tau}$ directly, one may actually use any of the available tests for cross-sectional dependence. We shall build on the familiar Breusch-Pagan [BP] test based on the sample correlations of all unique pairs $(u_{i,t,\tau}, u_{j,t,\tau})$, $i \neq j$.[2] Then, plugging in residuals for the unobserved regression errors is the natural way to proceed. The classical BP test resorts to LS residuals; here, however, one should rather employ QR residuals. This is because slope coefficients may well be quantile-specific, and we would thus take into account the fact that cross-sectional dependence may have different effects at different quantile levels. We consider pooled estimation first (allowing for fixed effects)

[2] In the above Gaussian example, the BP test is a Lagrange multiplier test, so we may argue in its favor using Gaussian quasi-likelihood arguments. Also, the BP test is readily implemented in many software packages.

and deal afterwards with slope parameter heterogeneity by means of individual-unit estimation. In fact, we do not focus on a particular choice of panel QR estimators, but rather require mild high-level assumptions on their convergence rates in a large-N large-T setup. This allows for a flexible use of the proposed test of no cross-dependence in panel QR practice.

We draw in the following on the large literature on tests for cross-sectional dependence in LS panels boosted by the seminal paper of Pesaran (2004). Some of the technical assumptions we make follow in fact this literature.

The proposed test statistic is constructed as follows:

1. Estimate a fixed-effects QR at the relevant quantile τ,

$$y_{i,t} = \hat{\alpha}_{i,\tau} + \hat{\boldsymbol{\beta}}_{\tau}' \boldsymbol{x}_{i,t} + \hat{u}_{i,t,\tau}.$$

2. Compute the pairwise correlation coefficients of the residual series,

$$\hat{\rho}_{ij,\tau} = \frac{\sum_{t=1}^{T} \left(\hat{u}_{i,t,\tau} - \bar{\hat{u}}_{i,\tau}\right)\left(\hat{u}_{j,t,\tau} - \bar{\hat{u}}_{j,\tau}\right)}{\sqrt{\sum_{t=1}^{T} \left(\hat{u}_{i,t,\tau} - \bar{\hat{u}}_{i,\tau}\right)^2 \sum_{t=1}^{T} \left(\hat{u}_{j,t,\tau} - \bar{\hat{u}}_{j,\tau}\right)^2}},$$

   where $\bar{\hat{u}}_{i,\tau} = T^{-1} \sum_{t=1}^{T} \hat{u}_{i,t,\tau}$.

   Given that – unlike fixed-effects LS residuals – the QR residuals $\hat{u}_{i,t,\tau}$ are not necessarily centered at zero, with the mean depending on the quantile level τ, unit-wise demeaning is necessary. This results in a slightly different statistic compared to the original BP test.

3. The test statistic is then given as,

$$\mathcal{T}_{\tau} = \frac{1}{\sqrt{N(N-1)}} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left(T\hat{\rho}_{ij,\tau}^2 - 1\right). \tag{3}$$

Since the BP-type statistic in (3) aggregates squared cross-correlations, the test rejects for large positive outcomes of $\mathcal{T}_{\tau}$. In the following, we show the limiting null distribution of $\mathcal{T}_{\tau}$ to be standard normal, regularity conditions provided:

Assumption 1 *Under the null hypothesis, the errors follow the multiplicative component structure* $u_{i,t,\tau} = \sigma_i \epsilon_{i,t}$*, where* σ_i *are positive constants bounded and bounded away from* 0*, and* $\epsilon_{i,t}$ *are independent of* $x_{i,t}$ *and iid across* i *and* t *with absolutely continuous pdf* f *and unity variance.*

The independence assumption for the errors under the null is quite common in the literature on testing for no cross-sectional dependence; see e.g. Baltagi et al. (2012). The dependence on the quantile level τ enters the model via quantile-specific regression coefficients. The continuity requirement for the pdf f is specific to the QR literature and allows, among others, for a characterization of the QR estimators. The assumption furthermore allows for error variance heterogeneity in the cross-sectional dimension. The unity variance requirement is a standard requirement for the standardized errors in the case of the BP test. While we do not pursue the topic of error variance heterogeneity in the time dimension here, we note that a robust version of the BP test following Halunga et al. (2017) may be considered instead of the classical form in (3).

The τ-quantile of the disturbances $u_{i,t,\tau}$ is given under the null hypothesis by $\sigma_i q_\tau$, with q_τ denoting the τ-quantile of $\epsilon_{i,t}$; as usual, this may be incorporated into the fixed effects α_i to ensure identification of the slope coefficients. Under cross-sectional dependence, we focus on sequences of local alternatives as follows.

Assumption 2 *Under the alternative hypothesis, let* $u_{i,t,\tau} = \sigma_i \epsilon_{i,t} + \boldsymbol{\lambda}'_{i,\tau} \boldsymbol{f}_t$*, where* $\frac{1}{T}\sum_{t=1}^{T} \boldsymbol{f}_t \boldsymbol{f}'_t \overset{p}{\to} \boldsymbol{\Sigma}_f > 0$ *as* $T \to \infty$ *and* $\boldsymbol{\lambda}_{i,\tau} = T^{-1/4} N^{-1/4} \boldsymbol{\ell}_{i,\tau}$*, with* $N^{-2} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \left(\boldsymbol{\ell}'_{i,\tau} \boldsymbol{\Sigma}_f \boldsymbol{\ell}_{j,\tau}\right)^2 \to c_\tau^2 < \infty$*.*

We note that such a local alternative corresponds to moderate cross-sectional dependence in the sense of Bailey et al. (2016). Furthermore, note that we consider local alternatives in $N^{-1/4}T^{-1/4}$-neighbourhoods of the null, and Section 2 argues that already such relatively weak cross-dependence may lead to panel QR bias. Of course, the rates for the local power follow from the structure of the test statistic. Finally, the loadings are not restricted to be homogeneous across different quantile levels.

Assumption 3 *The regressors $x_{i,t}$ have uniformly bounded 8th order moments, and satisfy $\frac{1}{T}\sum_{t=1}^{T}\left(\boldsymbol{x}_{i,t}-\bar{\boldsymbol{x}}_i\right)\left(\boldsymbol{x}_{i,t}-\bar{\boldsymbol{x}}_i\right)' \xrightarrow{p} \boldsymbol{\Sigma}_i$ uniformly in $i=1,\ldots,N$, with $\boldsymbol{\Sigma}_i$ positive definite matrices with eigenvalues uniformly bounded and bounded away from zero.*

For the pooled fixed-effects QR estimator, we only require a high level representation.

Assumption 4 *Let the following Bahadur-type representation hold under the null and the local alternative as $N,T\to\infty$,*

$$\sqrt{NT}\left(\hat{\boldsymbol{\beta}}_\tau-\boldsymbol{\beta}_\tau\right)=\left(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{\sigma_i}f(q_\tau)\boldsymbol{\Sigma}_i\right)^{-1}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\boldsymbol{x}_{i,t}-\bar{\boldsymbol{x}}_i\right)\psi_\tau\left(u_{i,t,\tau}-\sigma_i q_\tau\right)+R_{NT} \tag{4}$$

where $R_{NT}=O_p(1)$ and ψ_τ is the generalized sign function, $\psi_\tau(u)=\tau-\mathbb{I}(u<0)$ with $\mathbb{I}(\cdot)$ the usual indicator function.

This is the Bahadur representation for a linear model; see Kato et al. (2012), which is the consequence of the Assumptions 1 and 2. No explicit conditions at all are placed on the estimators of the fixed effects $\hat{\alpha}_{i,\tau}$; they are washed out from the cross-dependence statistic when demeaning the residuals $\hat{u}_{i,t,\tau}$. Of course, consistency of $\hat{\boldsymbol{\beta}}_\tau$ as implied by Assumption 4 is often related to the behaviour $\hat{\alpha}_{i,\tau}$ so in fact we do impose an implicit condition on the fixed-effects estimator.

Assumption 4 implies under the null – and in the local alternative setup – that $\hat{\boldsymbol{\beta}}_\tau-\boldsymbol{\beta}_\tau = O_p\left(1/\sqrt{NT}\right)$, where $\sqrt{NT}$ is the usual convergence rate of pooled or fixed-effects slope coefficient estimators. We note that R_{NT} in (4) need not be centered at zero, so estimators exhibiting 2nd order bias (as is the case in Remark 2) may be employed in our framework. Even so, the behavior of R_{NT} is not trivial (see Kato et al., 2012) and may require additional restrictions on N and T. We only formulate high-level assumptions here to allow for tractable analysis of the proposed cross-sectional dependence test.

We are now in a position to state the following proposition regarding the limit distribution of the test statistic in (3) under the null and the considered local alternatives.

Proposition 1 *Under Assumptions 1–4, as $N,T\to\infty$ with $N/T\to 0$, it holds that*

$$\mathcal{T}_\tau \xrightarrow{d} \mathcal{N}\left(c_\tau^2,1\right)$$

where c_τ^2 is as defined in Assumption 2.

Under the null ($c_\tau^2 = 0$), this collapses to the standard normal distribution and we may therefore reject the null hypothesis of no cross-sectional dependence at asymptotic size α if $\mathcal{T}_\tau$ exceeds the $1-\alpha$ quantile of the standard normal.

Plugging in estimates $\hat{u}_{i,t,\tau}$ for the unobserved $u_{i,t,\tau}$ has consequences on the finite-sample behaviour of the BP test if N is moderately large or large relative to T. This is in fact the case for LS residuals too, see e.g. Pesaran et al. (2008) and Baltagi et al. (2012). Since rate restrictions are difficult to check in practice, we suggest a finite-sample refinement based on an evaluation of vanishing components of $\mathcal{T}_\tau$. Concretely, it can be seen from the proof of Proposition 1 (see Appendix B) that most finite-sample distortions are induced by two asymptotically negligible terms (whose expectation is computed in the appendix), and we suggest the use of the corrected statistic,

$$\tilde{\mathcal{T}}_\tau = \mathcal{T}_\tau - \frac{\sqrt{N(N-1)}}{2T} - \frac{\tau(1-\tau)}{\hat{f}^2(q_\tau)} \frac{\sqrt{N(N-1)}}{T}. \tag{5}$$

The unknown density f of the standardized disturbances at the τth quantile may be estimated using the pooled standardized residuals under the null, $\hat{\epsilon}_{i,t} = \hat{u}_{i,t,\tau}/\hat{\sigma}_i$, where $\hat{\sigma}_i = \sqrt{T^{-1}\sum_{t=1}^{T}(\hat{u}_{i,t,\tau} - \bar{u}_{i,\tau})^2}$. Importantly, note that the residuals $\hat{u}_{i,t,\tau}$ have approximately zero τth quantile by construction, such that one should estimate their density *at zero*. In particular, we use a standard kernel density estimator [KDE] to this end. Also, consistency $\hat{\alpha}_{i,\tau}$ is required for this finite-sample correction. See Section 4 for recommendations on the choice of bandwidth.

Remark 4. Under the imposed rate restriction $N/T \to 0$, we have $\frac{\sqrt{N(N-1)}}{2T} \to 0$, such that $\tilde{\mathcal{T}}_\tau$ and $\mathcal{T}_\tau$ are asymptotically equivalent. The first term of the proposed correction is quite similar to that derived by Baltagi et al. (2012) for no error cross-correlation in a classical fixed-effects homogeneous panel data model, and essentially offsets terms that stem from demeaning the residuals. The second term is specific to the QR setup, and is designed to capture some of the level-specific effects of the slope coefficient estimation. $\diamondsuit$

If considering individual-unit estimation, we obtain the same limiting behavior if Assumption 4 is modified, as in Assumption 5 below, to allow for individual-unit QR estimation.

Assumption 5 *Let the following Bahadur representations hold as $N, T \to \infty$,*

$$\sqrt{T}\left(\hat{\boldsymbol{\beta}}_{i,\tau} - \boldsymbol{\beta}_{i,\tau}\right) = \left(\frac{1}{\sigma_i} f(q_\tau)\boldsymbol{\Sigma}_i\right)^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left(\boldsymbol{x}_{i,t} - \bar{\boldsymbol{x}}_i\right) \psi_\tau\left(u_{i,t,\tau} - \sigma_i q_\tau\right) + R_{iT} \tag{6}$$

where there exists $\delta > 0$ such that $\max_{1\le i\le N} \|R_{iT}\| = O_p\left(N^{(\delta+1)/2}\right)$.

We note that, given the moment restrictions on the regressors $\boldsymbol{x}$, Assumption 5 implies a *uniform* convergence rate of $O_p\left(\frac{N^{\delta/2}}{T^{1/2}}\right)$ for $\hat{\boldsymbol{\beta}}_{i,\tau}$; the *individual-unit* estimators $\hat{\boldsymbol{\beta}}_{i,\tau}$ may of course be $\sqrt{T}$-consistent.

The test statistic $\mathcal{T}_\tau$ is modified so that the residuals $\hat{u}_{i,t,\tau}$ are now obtained from individual regressions, that is,

$$\hat{u}^{(i)}_{i,t,\tau} = y_{i,t} - \hat{\alpha}_{i,\tau} - \hat{\boldsymbol{\beta}}'_{i,\tau}\boldsymbol{x}_{i,t}.$$

The following proposition states a trade-off between the uniform convergence rate of the unit-specific slope coefficient estimators (as characterized by δ in Assumption 5) and the dimensions of the panel: in a nutshell, the more estimation noise, the less cross-sectional units are allowed for in order to obtain a standard normal limiting distribution of the test statistics.

Proposition 2 *Under Assumptions 1–3 and 5, as $N, T \to \infty$ such that $\frac{N^{1+2\delta}}{T} \to 0$, it holds that*

$$\mathcal{T}^{(i)}_\tau \xrightarrow{d} \mathcal{N}\left(c^2_\tau, 1\right)$$

where c^2_τ is as defined in Assumption 2.

When $\delta = 0$ (which is in a sense closest to homogeneity in the unit-specific estimation setup), one recovers the $N = o(T)$ rate from Proposition 1.

The correction proposed in Eq. (5) may be used equally well for $\mathcal{T}^{(i)}_\tau$, and we denote the corrected statistic based on individual-estimation residuals by $\tilde{\mathcal{T}}^{(i)}_\tau$.

To conclude this section, we consider a simple portmanteau test for no cross-sectional dependence at several different quantiles, $\tau_1, \ldots, \tau_K$. We focus again on the statistics with finite-sample correction, and let $\tilde{\mathcal{T}}_{\tau_k}$ ($\tilde{\mathcal{T}}^{(i)}_{\tau_k}$) be the test statistics at quantile τ_k as in (5). Assume that either Assumption 4 or Assumption 5 holds at each of the K quantiles τ_k. The portmanteau statistic is then

$$\tilde{\mathcal{M}}_K = \frac{1}{K}\sum_{k=1}^{K}\tilde{\mathcal{T}}_{\tau_k} \tag{7}$$

($\tilde{\mathcal{M}}^{(i)}_K = \frac{1}{K}\sum_{k=1}^{K}\tilde{\mathcal{T}}^{(i)}_{\tau_k}$) and we again reject for test outcomes exceeding the $1-\alpha$ quantile of the standard normal distribution. Hence, the following proposition can be stated.

Proposition 3 *Under the Assumptions of either Propositions 1 or 2, it holds under the local alternative that*

$$\tilde{\mathcal{M}}_K \xrightarrow{d} \mathcal{N}\left(\bar{c}^2, 1\right)$$

and

$$\tilde{\mathcal{M}}^{(i)}_K \xrightarrow{d} \mathcal{N}\left(\bar{c}^2, 1\right),$$

respectively, where $\bar{c}^2 = \frac{1}{K}\sum_{k=1}^{K} c^2_{\tau_k}$ *with* $c^2_{\tau_k}$ *as defined in Assumption 2.*

4 Finite-sample evidence

Building on Pesaran et al. (2008) and Moscone and Tosetti (2009) we follow the setup of Demetrescu and Homm (2016) and use the following data generating process:

$$y_{i,t} = \alpha_i + \beta_1 x_{1,i,t} + \beta_2 x_{2,i,t} + u_{i,t,\tau}, \quad i = 1, \ldots, N, \text{ and } t = 1, ..., T \tag{8}$$

where $\beta_1 = \beta_2 = 1$. Moreover, we simulate regressors which, due to a factor structure, are correlated across cross-sections,

$$x_{l,i,t} = f^{(x)}_{l,t}\gamma^{(x)}_{l,i} + \epsilon^{(x)}_{l,i,t}$$

where $f_{l,t}^{(x)} \sim iid\, N(0,1)$ and $\epsilon_{l,i,t}^{(x)} \sim iid\, N(0,0.1)$. We set $\gamma_{l,i}^{(x)} = 1$, but one could also consider, for example, $\gamma_{l,i}^{(x)} \sim iid\, U(-0.2,0.2)$ with $U(a,b)$ standing for a uniform distribution on (a,b).[3] Further, we consider $\alpha_i \sim iid\mathcal{N}(1,1)$. The quantiles of interest are taken to be $\tau = \{0.2, 0.5, 0.8\}$.

We consider two scenarios for generating errors. First, we generate $u_{i,t,\tau}$ as,

$$u_{i,t,\tau} = \epsilon_{i,t} + \gamma_{1,\epsilon} f_{1,t}^{(x)} + \gamma_{2,\epsilon} f_{2,t}^{(x)},$$

where $\epsilon_{i,t} \sim iid\mathcal{N}(0,1)$ and independent from all the model variables so that we have homoskedastic idiosyncratic error terms (the difference between mean and quantile of interest is absorbed in the fixed-effect so centering at the relevant quantile is not necessary). Further, if $\gamma_{1,\epsilon} \neq 0$ or $\gamma_{2,\epsilon} \neq 0$, then we will have endogeneity which in turn induces the estimators of the model parameters to be biased. This serves to evaluate the test under the null hypothesis. Second, we consider

$$u_{i,t,\tau} = (\epsilon_{i,t} - z_\tau)\sqrt{1 + 0.5x_{1,i,t}^2 + 0.5x_{2,i,t}^2},$$

where z_τ is the τ-quantile of the standard normal distribution. Under the latter specification, $u_{i,t,\tau}$ is conditionally heteroskedastic, and dependent across the cross-sectional units, since $x_{l,i,t}$ are themselves dependent across the cross-sectional dimension. This serves to evaluate the proposed tests under the alternative.

The KDE of $f(q_\tau)$ is based on pooled normalized residuals, $\widehat{u}_{i,t,\tau}/\widehat{\sigma}_i$, where $\widehat{\sigma}_i$ is the standard deviation of $\{\widehat{u}_{i,t,\tau}\}_{t=1,2,\ldots,T}$. (Recall, the residual density should be estimated at zero and not at q_τ given the centering of the QR residuals.) We use a Gaussian kernel with a bandwidth of $0.35(NT)^{-0.2}$. The bandwidth is based on Silverman's rule of thumb, where we exploit the fact that the residuals are standardized prior to computing the KDE. Furthermore, it is smaller than the Silverman bandwidth choice for KDEs, which is due to

[3] This alternative design represents low regressor cross-dependence in the setup of Demetrescu and Homm (2016); however, this does not significantly change the results and we do not report them here.

the fact that the KDE of $f(q_\tau)$ is based here on residuals containing estimation noise, and a certain degree of undersmoothing was found in preliminary simulations to be beneficial to the finite-sample properties of the test.

We estimate the model unit-by-unit using the conventional QR procedure of Koenker and Bassett (1978), as well as in a pooled manner using the fixed-effects estimation procedure proposed by Koenker (2004). Results based on 2000 Monte Carlo replications for each case are given in Tables 1 and 4 for all quantiles τ of interest.

Table 1 provides the empirical rejection rates when the idiosyncratic error term is homoskedastic. As expected, the test based on $\mathcal{T}_\tau$ is oversized when T is relatively small, with distortions being somewhat larger for the individual-unit estimation case. This Table also shows that $\widetilde{\mathcal{T}}_\tau$ provides a good size correction for all quantiles of interest for almost all $\{N, T\}$ constellations for pooled estimation of the slope coefficients. Exceptions are observed when $\tau = 0.2$ and $\tau = 0.8$ with $N = 100$ and $T = 10$ where the rejection rate of $\widetilde{\mathcal{T}}_\tau$ turns out to be 8.1% and 8.3%, respectively. The resulting size control observed for the individual-unit estimation is effective in general too, but is sensitive to cases when N/T is bigger than 2. Further, when we observe size distortions for the individual-unit estimation, then these turn out to be larger when $\tau = 0.5$ compared to $\tau = 0.2$ and 0.8.

Table 1 also reports the rejection rates for the portmanteau statistic, $\widetilde{\mathcal{M}}_3$, which we calculate using the corrected statistic $\widetilde{\mathcal{T}}_\tau$ computed at the quantiles $\tau = \{0.2, 0.5, 0.8\}$. The observed behavior of $\widetilde{\mathcal{M}}_3$ is in line with that of the tests for individual quantiles.

Table 4 shows that the tests reject more often than under the previous scenario. This is not surprising since $u_{i,t,\tau}$ is cross-sectionally dependent through its dependence on $x_{i,t}$ (which is in turn cross-sectionally dependent). Also, the rejection frequencies increase as either N or T grow, apparently faster in N than in T. Both $\mathcal{T}_\tau$ and its corrected version $\widetilde{\mathcal{T}}_\tau$ are able to detect cross-sectional error dependence (where of course the corrected version should be preferred on the basis of the improved size control). The conclusions regarding the portmanteau statistic, $\widetilde{\mathcal{M}}_3$, are qualitatively the same. While the tests are, expectedly, not able to pin down the source of dependence, they are clearly indicative of misspecification. All in all, the tests appear to be a useful diagnostic tool for specifying panel QR models.

Table 1: Empirical rejection frequencies for $\mathcal{T}_\tau$ and $\widetilde{\mathcal{T}}_\tau$ under a homoskedastic error structure and no cross-unit error dependence with $\gamma_{1,\epsilon} = \gamma_{2,\epsilon} = 0$

		Individual-unit estimation							Pooled estimation						
T	N	$\mathcal{T}_{0.2}$	$\widetilde{\mathcal{T}}_{0.2}$	$\mathcal{T}_{0.5}$	$\widetilde{\mathcal{T}}_{0.5}$	$\mathcal{T}_{0.8}$	$\widetilde{\mathcal{T}}_{0.8}$	$\widetilde{\mathcal{M}}_3$	$\mathcal{T}_{0.2}$	$\widetilde{\mathcal{T}}_{0.2}$	$\mathcal{T}_{0.5}$	$\widetilde{\mathcal{T}}_{0.5}$	$\mathcal{T}_{0.8}$	$\widetilde{\mathcal{T}}_{0.8}$	$\widetilde{\mathcal{M}}_3$
10	10	14.0	6.6	20.6	11.4	13.5	6.5	5.8	12.0	5.5	12.5	5.4	11.7	5.8	5.3
20	10	8.4	5.1	10.2	7.1	9.2	6.0	5.0	8.3	5.4	7.7	5.3	8.2	5.3	4.9
30	10	6.6	3.9	8.5	5.5	7.0	4.8	4.0	6.9	5.0	7.4	5.0	7.8	5.6	5.2
50	10	7.0	5.2	7.6	5.8	6.6	5.0	5.1	6.8	5.2	6.9	5.2	6.8	5.0	5.2
100	10	7.0	6.1	6.9	6.1	6.8	6.0	5.9	6.9	5.7	6.5	5.6	6.7	5.7	5.6
10	20	30.5	7.4	52.3	19.0	31.0	7.1	7.6	27.1	5.6	26.7	5.5	25.8	6.1	5.4
20	20	13.2	5.2	19.0	7.7	12.6	4.8	4.4	12.7	4.9	12.8	5.2	12.9	4.9	5.0
30	20	11.2	6.1	12.4	6.7	10.6	5.2	5.3	11.2	5.3	11.1	5.1	11.0	5.1	5.0
50	20	9.2	4.9	10.3	6.2	8.6	4.8	4.9	9.2	5.1	9.2	4.9	9.2	4.9	4.8
100	20	6.5	4.8	7.2	5.4	6.6	4.7	4.8	6.7	4.5	6.5	4.7	7.0	4.6	4.4
10	30	54.9	9.2	82.6	29.7	54.0	7.8	11.3	48.9	6.1	48.1	5.8	48.2	6.5	6.0
20	30	18.3	4.9	28.3	9.3	18.1	4.8	4.8	18.8	4.5	18.3	4.8	18.1	5.3	4.8
30	30	12.2	4.0	16.8	6.0	11.5	3.9	3.7	11.0	3.8	11.2	3.8	11.2	3.6	3.6
50	30	9.2	4.7	11.6	5.6	9.3	4.3	4.6	9.6	4.6	9.8	4.9	9.7	4.9	4.8
100	30	6.8	4.3	7.7	4.4	7.4	3.9	4.2	6.3	4.0	6.3	4.0	6.1	3.7	3.9
10	50	93.5	12.9	99.9	52.7	92.9	11.9	19.8	88.7	6.3	89.1	6.3	89.1	6.5	6.1
20	50	36.6	3.6	56.5	10.7	35.7	3.7	4.0	34.4	3.2	35.1	3.2	35.1	3.2	3.0
30	50	20.4	4.0	31.8	7.4	20.9	4.0	4.3	20.6	3.7	20.8	3.7	21.0	3.7	3.7
50	50	12.2	3.4	15.3	4.4	11.4	2.9	3.4	12.0	2.9	12.0	2.8	11.4	2.7	2.7
100	50	7.8	3.1	9.6	3.5	8.3	3.3	3.1	8.6	2.9	9.0	3.1	8.8	3.0	2.8
10	100	100.0	23.8	100.0	93.0	100.0	24.2	56.4	100.0	8.8	100.0	7.9	100.0	8.5	8.6
20	100	84.6	3.7	98.1	22.0	85.7	3.4	5.8	85.1	3.5	84.8	3.4	85.0	3.5	3.4
30	100	50.6	2.6	74.3	8.8	49.2	3.2	3.8	49.3	2.3	49.4	2.3	50.0	2.4	2.4
50	100	25.9	2.4	38.2	4.7	24.8	2.5	2.8	25.6	2.3	25.5	2.2	25.2	2.3	2.3
100	100	12.2	2.5	15.7	3.3	12.4	2.1	2.7	11.7	2.8	11.8	2.8	11.8	2.8	2.9

Note: $\mathcal{T}_\tau$ and $\widetilde{\mathcal{T}}_\tau$, correspond to the test statistics in (3) and (5), respectively computed at quantiles $\tau = \{0.2, 0.5, 0.8\}$ using either individual-unit or pooled fixed-effects estimation. $\widetilde{\mathcal{M}}_3 = \frac{1}{3}(\widetilde{\mathcal{T}}_{0.2} + \widetilde{\mathcal{T}}_{0.5} + \widetilde{\mathcal{T}}_{0.8})$ corresponds to the portmanteau statistics in (7). All results reported are based on the nominal size of 5% and 2000 Monte Carlo replications.

Table 2: Empirical rejection frequencies for $\mathcal{T}_\tau$ and $\widetilde{\mathcal{T}}_\tau$ under a homoskedastic error structure and no cross-unit error dependence with $\gamma_{1,\epsilon} = \gamma_{2,\epsilon} = 0.5$

		Individual-unit estimation						Pooled estimation					
N	T	$\mathcal{T}_{0.2}$	$\widetilde{\mathcal{T}}_{0.2}$	$\mathcal{T}_{0.5}$	$\widetilde{\mathcal{T}}_{0.5}$	$\mathcal{T}_{0.8}$	$\widetilde{\mathcal{T}}_{0.8}$	$\mathcal{T}_{0.2}$	$\widetilde{\mathcal{T}}_{0.2}$	$\mathcal{T}_{0.5}$	$\widetilde{\mathcal{T}}_{0.5}$	$\mathcal{T}_{0.8}$	$\widetilde{\mathcal{T}}_{0.8}$
10	10	14.1	5.8	21.3	9.7	12.8	5.6	12.0	4.6	11.6	3.6	11.9	4.1
20	10	8.7	4.4	10.6	5.2	8.8	5.3	8.9	4.7	8.6	4.5	8.5	4.3
30	10	7.1	4.5	7.9	4.3	7.9	4.5	7.7	4.6	7.5	4.4	8.0	4.6
50	10	6.3	4.5	7.3	5.1	6.6	4.9	7.1	5.3	7.2	5.3	7.4	5.0
100	10	6.7	5.4	6.8	5.3	6.5	5.0	6.5	5.1	6.5	4.9	6.8	5.7
10	20	29.5	5.3	48.8	12.4	28.3	4.4	25.1	4.0	24.5	3.3	25.2	4.1
20	20	13.3	3.9	16.5	4.9	12.7	3.6	13.1	3.1	13.2	2.8	12.7	3.1
30	20	9.6	3.2	11.9	4.9	10.3	4.1	9.6	4.0	9.5	3.8	9.9	3.9
50	20	8.3	3.9	9.0	4.1	8.3	4.1	7.8	4.1	7.6	4.1	7.6	4.4
100	20	6.7	3.9	6.5	3.8	6.7	4.3	7.1	4.0	7.0	4.0	7.0	4.3
10	30	56.7	6.5	80.4	18.6	53.6	5.8	47.2	4.0	45.8	3.3	46.6	4.6
20	30	20.1	3.5	28.5	5.6	19.8	3.6	19.8	3.2	19.8	2.4	20.0	3.0
30	30	12.8	2.8	15.3	3.8	12.3	3.0	12.4	3.0	12.0	2.3	13.0	2.7
50	30	8.4	2.6	9.5	3.1	9.3	2.7	8.8	2.4	9.1	2.1	9.3	2.4
100	30	7.8	3.6	7.5	3.8	7.6	3.4	7.3	3.3	7.2	3.6	7.5	3.4
10	50	93.0	8.8	99.8	38.0	93.8	8.7	88.6	4.6	88.0	3.0	88.1	3.7
20	50	36.6	2.3	53.4	5.2	34.9	2.3	35.9	2.1	35.2	1.8	34.8	2.6
30	50	19.8	2.3	27.6	2.6	20.6	1.9	21.5	1.9	20.7	1.4	20.8	1.7
50	50	15.3	2.8	17.7	2.7	14.7	2.1	14.1	2.2	14.5	2.0	14.6	2.3
100	50	9.2	2.9	8.9	2.8	9.1	3.0	8.9	3.0	9.2	2.8	9.0	3.1
10	100	100.0	15.7	100.0	78.6	100.0	15.8	100.0	5.2	100.0	2.7	100.0	4.6
20	100	87.3	1.8	98.0	7.2	86.8	1.8	86.1	1.7	86.1	1.2	85.4	1.8
30	100	52.6	1.2	69.6	2.2	52.3	1.1	50.8	0.7	50.3	0.5	50.8	0.7
50	100	27.2	1.1	34.2	1.2	27.4	0.8	27.4	1.0	27.0	0.6	27.4	1.1
100	100	13.5	1.3	13.6	1.1	13.9	1.0	13.9	1.4	13.4	1.2	13.5	1.4

Note: $\mathcal{T}_\tau$ and $\widetilde{\mathcal{T}}_\tau$, correspond to the test statistics in (3) and (5), respectively computed at quantiles $\tau = \{0.2, 0.5, 0.8\}$ using either individual-unit or pooled fixed-effects estimation. $\widetilde{\mathcal{M}}_3 = \frac{1}{3}(\widetilde{\mathcal{T}}_{0.2} + \widetilde{\mathcal{T}}_{0.5} + \widetilde{\mathcal{T}}_{0.8})$ corresponds to the portmanteau statistics in (7). All results reported are based on the nominal size of 5% and 2000 Monte Carlo replications

Table 3: Empirical rejection frequencies for $\mathcal{T}_\tau$ and $\widetilde{\mathcal{T}}_\tau$ under a homoskedastic error structure and no cross-unit error dependence with $\gamma_{1,\epsilon} = \gamma_{2,\epsilon} = 1$

		Individual-unit estimation						Pooled estimation					
N	T	$\mathcal{T}_{0.2}$	$\widetilde{\mathcal{T}}_{0.2}$	$\mathcal{T}_{0.5}$	$\widetilde{\mathcal{T}}_{0.5}$	$\mathcal{T}_{0.8}$	$\widetilde{\mathcal{T}}_{0.8}$	$\mathcal{T}_{0.2}$	$\widetilde{\mathcal{T}}_{0.2}$	$\mathcal{T}_{0.5}$	$\widetilde{\mathcal{T}}_{0.5}$	$\mathcal{T}_{0.8}$	$\widetilde{\mathcal{T}}_{0.8}$
10	10	14.1	5.8	21.3	9.7	12.8	5.6	12.0	4.6	11.6	3.6	11.9	4.1
20	10	8.7	4.4	10.6	5.2	8.8	5.3	8.9	4.7	8.6	4.5	8.5	4.3
30	10	7.1	4.5	7.9	4.3	7.9	4.5	7.7	4.6	7.5	4.4	8.0	4.6
50	10	6.3	4.5	7.3	5.1	6.6	4.9	7.1	5.3	7.2	5.3	7.4	5.0
100	10	6.7	5.4	6.8	5.3	6.5	5.0	6.5	5.1	6.5	4.9	6.8	5.7
10	10	14.0	6.7	19.7	9.0	14.5	6.0	12.8	5.0	11.9	4.1	12.1	4.5
20	10	8.9	4.6	10.1	5.5	9.2	5.5	9.7	4.8	8.2	4.4	8.7	4.7
30	10	8.2	5.5	7.5	4.4	9.0	5.0	8.5	5.7	8.1	5.2	8.6	5.5
50	10	7.2	4.8	7.2	4.7	7.7	5.0	7.7	5.3	7.2	5.0	7.8	5.3
100	10	7.6	6.6	7.2	5.9	7.7	6.3	7.5	6.2	7.2	6.1	8.4	7.2
10	20	31.0	6.2	46.4	11.2	29.9	5.7	26.2	4.7	25.0	3.8	25.6	4.6
20	20	14.2	4.6	16.4	4.7	14.0	4.2	14.4	4.2	13.5	3.4	14.0	4.1
30	20	11.8	4.7	11.2	4.6	11.5	4.7	10.8	4.8	10.2	4.0	11.0	4.8
50	20	10.5	5.8	8.9	4.6	9.6	4.9	10.2	5.6	9.0	4.6	10.1	5.1
100	20	10.0	5.9	8.6	4.9	9.8	5.7	10.0	6.6	9.2	5.7	10.3	6.3
10	30	56.2	7.5	76.5	15.9	54.8	7.0	49.1	4.9	47.1	3.5	47.5	5.0
20	30	21.5	3.6	25.0	4.6	22.4	3.7	21.1	3.9	20.4	3.1	21.0	3.6
30	30	14.9	3.6	14.3	3.4	13.8	3.9	15.2	4.1	13.9	3.9	14.4	4.2
50	30	12.1	3.7	10.6	3.5	11.5	3.7	11.5	4.5	10.6	3.6	12.2	4.3
100	30	11.6	6.1	9.5	4.5	12.2	6.9	12.0	6.1	11.3	5.7	12.2	6.5
10	50	93.4	10.8	99.4	30.7	93.4	9.7	88.7	5.5	88.5	3.1	88.4	4.7
20	50	41.0	3.2	48.4	3.6	39.6	3.2	38.5	3.4	37.7	2.5	37.9	3.6
30	50	25.3	3.5	24.5	2.3	25.7	2.8	26.3	2.5	25.1	2.0	25.3	2.7
50	50	21.9	4.8	16.7	2.6	20.0	3.8	20.1	4.6	20.1	3.7	21.1	4.6
100	50	20.5	7.4	13.6	4.7	20.6	7.9	20.2	6.9	18.9	6.7	20.0	8.0
10	100	100.0	19.2	100.0	68.0	100.0	19.8	100.0	8.0	100.0	3.9	100.0	6.8
20	100	90.3	3.6	96.1	4.7	89.8	3.9	88.6	3.5	88.2	1.9	87.9	3.4
30	100	63.5	3.0	64.8	1.2	61.7	2.9	63.5	2.5	61.9	1.5	61.9	2.0
50	100	45.1	3.7	35.6	1.0	45.1	3.7	44.5	4.0	44.9	2.8	45.3	3.5
100	100	41.7	11.2	26.9	4.6	41.0	10.7	41.7	10.8	40.5	9.5	40.8	10.0

Note: $\mathcal{T}_\tau$ and $\widetilde{\mathcal{T}}_\tau$, correspond to the test statistics in (3) and (5), respectively computed at quantiles $\tau = \{0.2, 0.5, 0.8\}$ using either individual-unit or pooled fixed-effects estimation. $\widetilde{\mathcal{M}}_3 = \frac{1}{3}(\widetilde{\mathcal{T}}_{0.2} + \widetilde{\mathcal{T}}_{0.5} + \widetilde{\mathcal{T}}_{0.8})$ corresponds to the portmanteau statistics in (7). All results reported are based on the nominal size of 5% and 2000 Monte Carlo replications

Table 4: Empirical rejection frequencies for $\mathcal{T}_\tau$ and $\widetilde{\mathcal{T}}_\tau$ under a heteroskedastic error structure and no cross-unit error dependence

		Individual-unit estimation							Pooled estimation						
T	N	$\mathcal{T}_{0.2}$	$\widetilde{\mathcal{T}}_{0.2}$	$\mathcal{T}_{0.5}$	$\widetilde{\mathcal{T}}_{0.5}$	$\mathcal{T}_{0.8}$	$\widetilde{\mathcal{T}}_{0.8}$	$\widetilde{\mathcal{M}}_3$	$\mathcal{T}_{0.2}$	$\widetilde{\mathcal{T}}_{0.2}$	$\mathcal{T}_{0.5}$	$\widetilde{\mathcal{T}}_{0.5}$	$\mathcal{T}_{0.8}$	$\widetilde{\mathcal{T}}_{0.8}$	$\widetilde{\mathcal{M}}_3$
10	10	20.4	12.2	24.7	14.4	21.3	11.6	9.5	27.4	17.0	24.0	13.9	27.0	16.6	14.2
20	10	23.8	17.2	23.1	16.8	24.8	17.5	14.7	30.1	22.5	26.4	19.5	31.4	24.2	21.1
30	10	27.4	22.2	23.6	18.7	25.7	20.5	18.8	31.5	26.0	26.4	21.2	31.0	26.2	23.5
50	10	30.4	26.6	24.6	21.2	29.8	26.2	22.7	33.8	29.8	26.3	22.4	33.7	29.6	27.3
100	10	40.4	38.2	27.7	24.7	40.8	38.1	33.3	42.9	40.2	28.2	25.9	42.7	39.9	35.4
10	20	50.1	21.5	60.7	28.9	51.6	21.3	19.6	63.2	33.2	60.0	27.4	63.6	31.9	30.6
20	20	48.6	31.8	48.8	30.8	47.8	31.7	30.8	58.8	43.4	54.5	37.6	61.4	45.2	42.4
30	20	56.1	44.1	49.8	36.6	53.4	40.5	41.4	65.6	52.9	55.1	42.9	62.0	51.5	49.7
50	20	65.4	58.0	56.6	46.9	66.3	57.8	55.8	72.0	63.1	60.0	51.5	72.1	64.8	61.5
100	20	80.6	76.1	60.2	54.7	78.2	74.3	72.4	82.0	77.4	61.9	56.9	81.2	77.1	74.2
10	30	77.1	31.8	87.8	42.4	77.9	30.9	32.8	87.7	47.6	85.3	42.2	87.5	46.0	45.6
20	30	72.4	48.9	71.3	47.9	72.8	49.0	49.1	81.9	63.3	77.8	54.8	81.4	63.2	61.4
30	30	79.9	65.6	76.0	58.9	79.5	65.1	65.7	85.6	73.9	80.8	66.3	85.3	74.5	73.0
50	30	88.0	79.9	79.9	68.7	87.7	80.3	79.5	90.4	84.7	83.8	74.5	90.4	84.9	83.7
100	30	95.0	93.0	84.4	78.3	95.7	93.3	92.7	95.8	93.8	86.0	80.9	96.8	94.2	93.4
10	50	99.1	52.4	99.9	69.4	99.1	52.2	60.5	99.5	68.9	99.6	65.3	99.4	69.3	69.1
20	50	94.6	73.8	96.3	73.6	94.9	73.0	76.5	97.8	85.6	97.3	81.3	97.6	85.0	85.3
30	50	96.6	87.2	95.9	84.6	96.9	87.6	88.7	98.2	92.3	97.7	87.8	98.3	92.7	92.2
50	50	99.1	96.6	98.1	92.6	99.0	96.4	97.0	99.5	98.0	98.6	94.8	99.5	97.9	97.9
100	50	100.0	99.9	99.0	97.0	99.9	99.6	99.8	100.0	100.0	99.1	97.9	99.8	99.7	99.8
10	100	100.0	84.4	100.0	97.7	100.0	84.6	93.8	100.0	91.2	100.0	90.1	100.0	91.2	91.8
20	100	100.0	95.9	100.0	97.3	99.9	95.6	97.1	100.0	98.7	100.0	97.7	100.0	98.3	98.3
30	100	100.0	99.1	100.0	99.0	100.0	99.2	99.0	100.0	99.5	100.0	99.2	100.0	99.5	99.4
50	100	100.0	99.9	100.0	99.9	100.0	99.8	99.9	100.0	100.0	100.0	99.9	100.0	100.0	100.0
100	100	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

Note: $\mathcal{T}_\tau$ and $\widetilde{\mathcal{T}}_\tau$, correspond to the test statistics in (3) and (5), respectively computed at quantiles $\tau = \{0.2, 0.5, 0.8\}$ using either individual-unit or pooled fixed-effects estimation. $\widetilde{\mathcal{M}}_3 = \frac{1}{3}(\widetilde{\mathcal{T}}_{0.2} + \widetilde{\mathcal{T}}_{0.5} + \widetilde{\mathcal{T}}_{0.8})$ corresponds to the portmanteau statistics in (7). All results reported are based on the nominal size of 5% and 2000 Monte Carlo replications.

5 A panel QR analysis of housing market growth

Homes are one of the most important assets in many households' portfolios (Englund et al., 2002) and, consequently, changes in housing wealth may lead to changes in homeowners' consumption (Case et al., 2005). E.g., it has been shown that the impact of changes in housing wealth on the economy can be more important than changes in wealth caused by stock price movements (Helbling and Terrones, 2003, and Rapach and Strauss, 2006). Economic history indeed suggests that some of the most severe systemic financial crises have been associated with boom-bust cycles in real estate markets (see e.g. Bordo and Jeanne, 2002, and Crowe et al., 2013).

In this context, Deghi et al. (2020) propose the so-called houses-prices-at-risk approach as a measure to evaluate risks to the real estate market. This measure is inspired in the work of Adrian et al. (2022) (see also Adrian et al., 2019) who developed a measure to evaluate risks to GDP growth (Growth-at-Risk); see Brownlees and Souza (2021) and Nandi (2022) for panel approaches to Growth-at-Risk. In a similar vein, Makabe and Norimasa (2022) analyse the term structure of Inflation-at-Risk. Such approaches estimate a (panel) QR to determine which of the covariates considered affect the response variable of interest, i.e. house price growth (for houses-prices-at-risk), inflation (for inflation-at-risk), or GDP growth (for growth-at-risk) and to explain the conditional predictive distribution of the response variable derived from the estimates. Moreover, the entire conditional distribution of the variable of interest is computed following two steps: (1) panel QR estimation of the effect of the covariates at each quantile, and (2) approximation of the estimated quantile function e.g. with a skewed t-distribution. Consequently, the correct estimation in the first step is of tantamount importance in this approach. In this section, we illustrate the relevance of our procedure with an application of panel QR to house price growth data for eleven countries.

5.1 Data

In our analysis we consider a balanced panel of quarterly time series, for the period from 1995:Q1 to 2020:Q3 ($T = 103$), for nine Euro Area countries (Germany (DE), France (FR), Italy (IT), Spain (ES), the Netherlands (NL), Ireland (IE), Portugal (PT), Belgium (BE) and Finland (FI)), the UK and the US ($N = 11$). Data on house prices, disposable income, labour force and private consumption deflator were collected from the OECD, while short-term interest rates were taken from the European Central Bank. A detailed description of all data sources and availability, as well as country specificities are provided in Appendix C.

House price indices correspond generally to seasonally unadjusted series constructed from national data from a variety of public and/or private sources (e.g., national statistical services, mortgage lenders and real estate agents). National house price series may differ in terms of dwelling types and geographical coverage (most are country-wide and refer to existing apartments). Several series are based on hedonic approaches to price measurement, characterized by valuing the houses in terms of their attributes (average square meter price, size of the dwellings involved in transactions and their location).

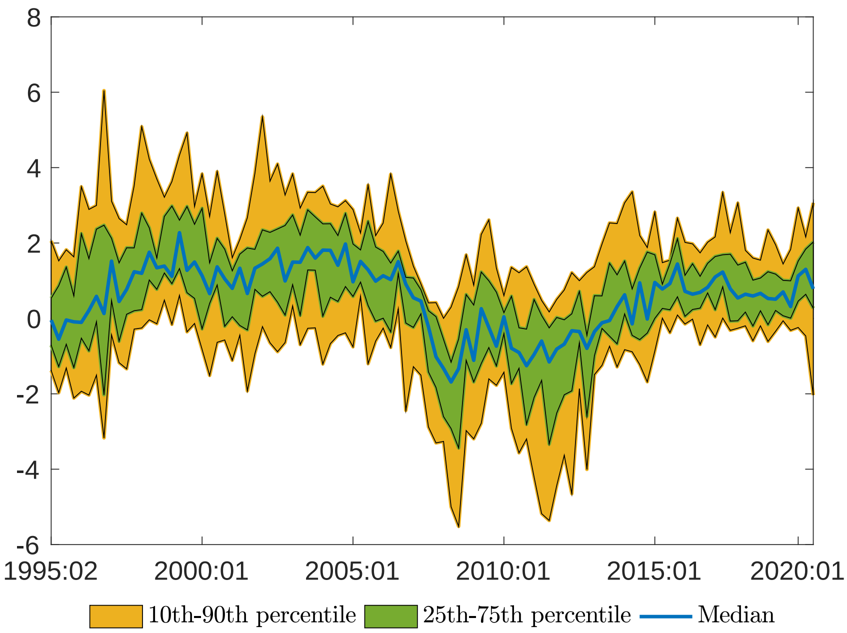

Figure 1: Quarterly change in log real house prices (in percentage) for 11 quarterly real house price series

In our analysis we consider fluctuations in real house prices,[4] measured as quarterly changes in the natural logarithm of the real house price index of each country, i.e., quarterly real house price growth. Figure 1 plots the cross-sectional 10th-90th percentile range, the 25th-75th percentile range and the median of the 11 quarterly real house price growth at each time in the sample.

This figure illustrates that, although some countries appear to be more cyclical than others, real house prices tend to co-move during crises, which suggests the presence of an underlying common factor in these series. We see a general decline during the global financial crisis (2008-2009) as well as during the European sovereign debt crisis (2011-2012).

5.2 Model

There is a vast number of studies that analyses the determinants of house prices and their growth. Findings in the literature indicate that models that explain changes in house prices include a wide set of fundamentals, such as income (or GDP), population, employment or unemployment rate, taxes, borrowing costs, construction costs and returns on alternative assets (Poterba et al., 1991, Englund and Ioannides, 1997, Tsatsaronis and Zhu, 2004).

In our analysis, the dependent variable is the growth rate of real house prices, Δrhp. To keep the model tractable, and due to data availability, we focus on the most consensual fundamentals, such as, log of real disposable income, $lrdi$, real mortgage interest rate, $rmtgr$, log of gross fixed capital formation, $lGFCF$, the unemployment rate, $unemp$, and the volume of loans for house purchases, $vlhp$.

We take a predictive perspective here, and the panel QR model is given as

$$\begin{aligned} \Delta rhp_{i,t} &= \alpha_{i,\tau} + \beta_{1,\tau}\Delta lrdi_{i,t-1} + \beta_{2,\tau}\Delta lGFCF_{i,t-1} + \beta_{3,\tau}\Delta vlhp_{i,t-1} \\ &\quad + \beta_{4,\tau} unemp_{i,t-1} + \beta_{5,\tau} rmtgr_{i,t-1} + \boldsymbol{\lambda}'_{i,\tau}\boldsymbol{f}_{t,\tau} + u_{i,t,\tau}, \end{aligned} \tag{9}$$

[4] All series in real terms are computed using the private consumption deflator.

where $\tau \in (0, 1)$ is the quantile of interest, $i = 1, \ldots, 11$ indexes the eleven countries considered, and Δ is the first difference operator.

The quantile factor methodology recently proposed by Chen et al. (2021), which we use to estimate a panel QR model with factors in (9), allows for quantile-dependent factors, $\boldsymbol{f}_{t,\tau}$. Our tests should detect such forms of cross-sectional dependence as well. The number of factors considered at each quantile is determined using the rank-minimization approach proposed by Chen et al..

Table 5: Panel QR results from models with and without quantile factors (QR_F and QR_0, respectively)

	QR_0	QR_F	QR_0	QR_F	QR_0	QR_F
	$\tau = 0.1$		$\tau = 0.2$		$\tau = 0.3$	
$\beta_{1,\tau}$	0.2293^{***}	0.1554^{***}	0.1136^{***}	0.0638^{***}	0.1174^{***}	0.0695^{***}
$\beta_{2,\tau}$	0.1072^{***}	0.0543^{***}	0.1220^{***}	0.0816^{***}	0.1175^{***}	0.0759^{***}
$\beta_{3,\tau}$	0.2260^{***}	0.1305^{***}	0.2205^{***}	0.0896^{***}	0.1909^{***}	0.1007^{***}
$\beta_{4,\tau}$	-0.1258^{***}	-0.0555^{***}	-0.0807^{***}	-0.0749^{***}	-0.0537^{***}	-0.0428^{***}
$\beta_{5,\tau}$	-0.1597^{***}	-0.0733^{***}	-0.1295^{***}	-0.0486^{**}	-0.0922^{***}	-0.0683^{***}
$f_{1,\tau}$		-0.0037^{*}		-0.0038^{**}		0.0017
$f_{2,\tau}$						0.0091^{***}
	$\tau = 0.4$		$\tau = 0.5$		$\tau = 0.6$	
$\beta_{1,\tau}$	0.1131^{***}	0.0648^{***}	0.0669^{**}	0.0441	0.0469	0.0498^{*}
$\beta_{2,\tau}$	0.0968^{***}	0.0756^{***}	0.1017^{***}	0.0694^{***}	0.0817^{***}	0.0586^{***}
$\beta_{3,\tau}$	0.1955^{***}	0.0827^{***}	0.1904^{***}	0.0838^{***}	0.1652^{***}	0.0495^{**}
$\beta_{4,\tau}$	-0.0439^{***}	-0.0474^{***}	-0.0363^{**}	-0.0331^{**}	-0.0246^{*}	0.0137
$\beta_{5,\tau}$	-0.0718^{***}	-0.1081^{***}	-0.0607^{***}	-0.0974^{***}	-0.0638^{***}	-0.0900^{***}
$f_{1,\tau}$		0.0059^{***}		0.0072^{***}		0.0088^{***}
$f_{2,\tau}$		-0.0123^{***}		-0.0114^{***}		0.0116^{***}
$f_{3,\tau}$		-0.0056^{***}		-0.0046^{**}		0.0034^{*}
$f_{4,\tau}$		0.0143^{***}		0.0138^{***}		0.0113^{***}
$f_{5,\tau}$				0.0107^{***}		0.0114^{***}
	$\tau = 0.7$		$\tau = 0.8$		$\tau = 0.9$	
$\beta_{1,\tau}$	0.0472	0.0454^{*}	0.0416	0.0479	0.0494	0.0379
$\beta_{2,\tau}$	0.0693^{***}	0.0552^{***}	0.0527^{***}	0.0465^{***}	0.0268	0.0531^{***}
$\beta_{3,\tau}$	0.1471^{***}	0.0560^{***}	0.1485^{***}	0.0351^{*}	0.1438^{***}	0.0032
$\beta_{4,\tau}$	-0.0356^{**}	0.0088	-0.0170	-0.0053	0.0119	-0.0179
$\beta_{5,\tau}$	-0.0692^{***}	-0.1153^{***}	-0.0472^{**}	-0.0902^{***}	-0.0366	0.0066
$f_{1,\tau}$		0.0094^{***}		-0.0072^{***}		0.0058^{***}
$f_{2,\tau}$		0.0107^{***}				

Note: Quantile regression estimation results of (9) with (QR_F) and without (QR_0) the inclusion of factors. The factors used where extracted using the approach of Chen et al. (2021).

Table 5 provides the estimation results of the panel QR model in (9) with (QR_F) and without (QR_0) the inclusion of factors.

The signs of the parameter estimates in Table 5 are in general as expected. Specifically, positive variations in the log of real disposable income, *lrdi* ($\beta_{1,\tau}$), the log of gross fixed capital formation, $lGFCF$ ($\beta_{2,\tau}$) and the volume of loans for house purchases, *vlhp* ($\beta_{3,\tau}$) have positive impacts on house price growth whereas positive variations in the unemployment rate, *unemp* ($\beta_{4,\tau}$), and the real mortgage interest rate, *rmtgr* ($\beta_{5,\tau}$), have negative impacts on house price growth. Moreover, we also observe that the association between the covariates and house price growth varies at the different parts of the house price growth distribution. Overall, the differences in slopes indicate a markedly stronger relationship towards the left tail of the future house prices growth distribution relatively to the median and the upper percentiles of the distribution.

Importantly, the QR_F estimation results highlight the relevance of the quantile factors used in the panel QR model. This Table shows that the factors are in general all statistically significant regardless of the quantile τ considered. Furthermore, if we contrast the slope parameter estimates obtained from QR_0 and QR_F we observe that the slope estimates are in general different.[5]

To formally support the choice of the QR_F results, Table 6 provides the outcomes of the QR cross-sectional dependence tests introduced here at quantiles $\tau \in \{0.1, 0.2, \ldots, 0.9\}$.

In addition to the results in Table 6 we have also computed the classical Breuch-Pagan test, $BP = 31.144$, and the bias-corrected version proposed by Baltagi et al. (2012), $BP_{bc} = 31.089$.

The results in Table 6 indicate that:

1. there is not a significant difference between the asymptotic and the corrected versions of the panel QR cross-sectional dependence tests;

2. the strength of the cross-correlation depends to some extent on the quantile of interest. The BP and the BP_{bc} tests do not provide quantile specific information.

[5]This is also observed by Nandi (2022) when explicitly accounting for cross-unit dependence in the panel QR analysis of Brownlees and Souza (2021).

3. there are visible differences between the tests based on pooled estimation ($\mathcal{T}_\tau$ and $\widetilde{\mathcal{T}}_\tau$) and those based on individual-unit estimation ($\mathcal{T}_\tau^{(i)}$ and $\widetilde{\mathcal{T}}_\tau^{(i)}$), where the latter indicates stronger cross-correlation. This points towards heterogeneity of the slope parameters in addition to cross-unit error dependence.

Table 6: Cross-sectional dependence test results

τ	$\mathcal{T}_\tau^{(i)}$	$\widetilde{\mathcal{T}}_\tau^{(i)}$	$\mathcal{T}_\tau$	$\widetilde{\mathcal{T}}_\tau$
0.100	16.244	16.151	25.583	25.489
0.200	17.613	17.515	25.853	25.754
0.300	15.369	15.269	27.721	27.622
0.400	17.419	17.316	28.122	28.019
0.500	19.539	19.441	28.899	28.801
0.600	19.341	19.243	33.287	33.188
0.700	21.424	21.326	35.864	35.766
0.800	20.687	20.592	37.298	37.202
0.900	30.436	30.337	41.305	41.207

Note: $\mathcal{T}_k$ and $\widetilde{\mathcal{T}}_k$ are the test statistics provided in (3) and (5), respectively; and $\mathcal{T}_k^{(i)}$ and $\widetilde{\mathcal{T}}_k^{(i)}$ are also computed as indicated in (3) and (5), respectively, but the residuals used are obtained from individual regressions.

Hence, overall Table 6 points to the presence of cross-sectional dependence which suggests that this feature needs to be addressed in the panel QR estimation and hence, supports the results obtained from the factor augmented panel QR model in (9).

Since Table 6 is suggestive of slope coefficient heterogeneity, we provide individual-estimation results in Appendix C (Table C2) and we also present plots of the country specific quantile predictions (Figure C1). Interestingly, during the COVID 19 pandemic the development of the housing market has been atypical. This is, to a certain extend, well illustrated in Figure C1. Specifically, we note that at the end of the sample, for many of the countries considered, the covariates point to an evolution of house price growth which is in contrast to the actual observed house price growth dynamics. In past recessions, downturns were typically followed by a moderate fall in nominal house prices, lasting about four quarters. However, in the pandemic period until the end of 2021, there was no decline at all. In addition, the current recession has not been accompanied by significant changes in credit growth, unlike past episodes, when households typically reduced their leverage after it had increased in the expansion phase (Igan et al., 2022).

In recent years, the international synchronization of house prices has increased. As noted by Igan et al. (2022), more than 60% of house price movements can now be explained by a common global factor. One reason for this much higher synchronization is that the pandemic has been truly global, thus inducing similar policy reactions and flattening yield curves worldwide.

6 Concluding remarks

This paper has argued that cross-sectional dependence in panel QR models may have a biasing effect on the QR estimator even if the latent error common components are independent of the regressors. This extends more generally to panel nonlinear GMM estimators with errors having a factor structure.

Motivated by this argument, we proposed a test for no cross-sectional dependence. Such tests may also be interpreted as misspecification tests, since the detection of cross-sectional dependence may imply the existence of potential estimation biases.

The proposed test is a version of the familiar Breusch-Pagan test based on residuals from either pooled or individual-unit QR estimation. While it possesses a standard normal limiting distribution under joint N, T asymptotics, the rate restrictions are not benign, which is reflected in the finite-sample behavior. For this reason we discuss a finite-sample correction which largely removes the size distortions when N is too large in relation to T. We also discuss a portmanteau version of the tests which aggregates evidence across several quantiles. Moreover, we provide an in-depth Monte Carlo analysis of the finite sample size and power properties of the new procedures introduced, confirming the usefulness of the finite-sample correction and revealing interesting power performance under the alternative.

Finally, we illustrate the usefulness of our approach in an empirical analysis of house-price growth determinants, from a predictive perspective, in a panel of eleven countries (Germany (DE), France (FR), Italy (IT), Spain (ES), the Netherlands (NL), Ireland (IE), Portugal (PT), Belgium (BE) and Finland (FI), the UK and the US), for the period from 1995:Q1 to 2020:Q3. The tests introduced clearly highlight the need to address cross-sectional dependence, favoring therefore a factor augmented panel QR model. Furthermore, evidence of cross-dependence is stronger in pooled residuals than in residuals from individual-unit estimation, indicating the presence of slope coefficient heterogeneity in addition to cross-unit dependence in the data.

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Appendix A - Auxiliary results

Throughout the appendix, let $\boldsymbol{u}_i$, $\widehat{\boldsymbol{u}}_{i,\tau}$ and $\mathbf{X}_i$ stack $u_{i,t,\tau}$, $\hat{u}_{i,t,\tau}$ and $\boldsymbol{x}'_{i,t}$ for $t = 1, ..., T$, and denote by $\widehat{\sigma}_{ij}$ the sample covariance of the residuals, $\widehat{\sigma}_{ij} = \frac{1}{T}\left(\widehat{\boldsymbol{u}}_{i,\tau} - \bar{\hat{u}}_{i,\tau}\boldsymbol{\iota}\right)'\left(\widehat{\boldsymbol{u}}_{j,\tau} - \bar{\hat{u}}_{j,\tau}\boldsymbol{\iota}\right)$ with $\boldsymbol{\iota}$ a T-vector of ones, and by $\ddot{\boldsymbol{a}}_i$ ($\ddot{\mathbf{A}}_i$) the column-specific demeaning of a vector (matrix).

Lemma 1 *Under the weaker assumptions of Proposition 2*

1. $Q_1 = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \frac{T}{\sigma_{ii}^2\sigma_{jj}^2}\left(\frac{1}{T}\left(\hat{\boldsymbol{\beta}}_{i,\tau} - \boldsymbol{\beta}_i\right)'\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j\right)^2 = o_p(N);$
2. $Q_2 = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \frac{T}{\sigma_{ii}^2\sigma_{jj}^2}\left(\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_{i,\tau} - \boldsymbol{\beta}_i\right)\right)^2 = o_p(N);$
3. $Q_3 = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} T\frac{\left(\frac{1}{T}\left(\hat{\boldsymbol{\beta}}_{i,\tau}-\boldsymbol{\beta}_i\right)'\ddot{\mathbf{X}}_i'\mathbf{X}_j\left(\hat{\boldsymbol{\beta}}_{i,\tau}-\boldsymbol{\beta}_i\right)\right)^2}{\sigma_{ii}^2\sigma_{jj}^2} = o_p(N);$
4. $Q_4 = -2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \frac{\ddot{\mathbf{u}}_i'\ddot{\mathbf{u}}_j\left(\hat{\boldsymbol{\beta}}_{i,\tau}-\boldsymbol{\beta}_i\right)'\ddot{\mathbf{X}}_i'\ddot{\mathbf{u}}_j}{T\sigma_{ii}^2\sigma_{jj}^2} = o_p(N);$
5. $Q_5 = -2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \frac{\ddot{\mathbf{u}}_i'\ddot{\mathbf{u}}_j\ddot{\mathbf{u}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_{i,\tau}-\boldsymbol{\beta}_i\right)'}{T\sigma_{ii}^2\sigma_{jj}^2} = o_p(N);$
6. $Q_6 = 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \frac{\ddot{\mathbf{u}}_i'\ddot{\mathbf{u}}_j\left(\hat{\boldsymbol{\beta}}_{i,\tau}-\boldsymbol{\beta}_i\right)'\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_{i,\tau}-\boldsymbol{\beta}_i\right)}{T\sigma_{ii}^2\sigma_{jj}^2} = o_p(N);$
7. $Q_7 = 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \frac{\left(\hat{\boldsymbol{\beta}}_{i,\tau}-\boldsymbol{\beta}_i\right)'\ddot{\mathbf{X}}_i'\ddot{\mathbf{u}}_j\ddot{\mathbf{u}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_{i,\tau}-\boldsymbol{\beta}_i\right)}{T\sigma_{ii}^2\sigma_{jj}^2} = o_p(N);$
8. $Q_8 = -2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \frac{\left(\hat{\boldsymbol{\beta}}_{i,\tau}-\boldsymbol{\beta}_i\right)'\ddot{\mathbf{X}}_i'\ddot{\mathbf{u}}_j\left(\hat{\boldsymbol{\beta}}_{i,\tau}-\boldsymbol{\beta}_i\right)'\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_{i,\tau}-\boldsymbol{\beta}_i\right)}{T\sigma_{ii}^2\sigma_{jj}^2} = o_p(N);$
9. $Q_9 = -2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \frac{\ddot{\mathbf{u}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_{i,\tau}-\boldsymbol{\beta}_i\right)\left(\hat{\boldsymbol{\beta}}_{i,\tau}-\boldsymbol{\beta}_i\right)'\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_{i,\tau}-\boldsymbol{\beta}_i\right)}{T\sigma_{ii}^2\sigma_{jj}^2} = o_p(N);$
10. $Q_{10} = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} T\frac{\widehat{\sigma}_{ij}^2\left(1-\frac{\widehat{\sigma}_{ii}^2\widehat{\sigma}_{jj}^2}{\sigma_{ii}^2\sigma_{jj}^2}\right)}{\widehat{\sigma}_{ii}^2\widehat{\sigma}_{jj}^2} = o_p(N),$

where Q_1 through Q_{10} are computed using either a pooled slope coefficient estimator or individual-unit estimators.

Lemma 2 *Under the weaker assumptions of Proposition 2, let $\alpha_T = \alpha_N = \frac{1}{4}$. Then,*

1. $S_1 = \frac{1}{\sqrt{N(N-1)}}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{1}{T^{1/2+2\alpha_T}N^{2\alpha_N}}\boldsymbol{\ell}_{i,\tau}'\bar{\mathbf{F}}'\bar{\mathbf{F}}\boldsymbol{\ell}_{j,\tau}\right)^2 = c_\tau^2 + o_p(1);$
2. $S_2 = \frac{1}{\sqrt{N(N-1)}}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{\sigma_i}{T^{1/2+\alpha_T}N^{\alpha_N}}\boldsymbol{\epsilon}_i'\bar{\mathbf{F}}\boldsymbol{\ell}_{j,\tau}\right)^2 = o_p(1);$
3. $S_3 = \frac{1}{\sqrt{N(N-1)}}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{\sigma_j}{T^{1/2+\alpha_T}N^{\alpha_N}}\boldsymbol{\ell}_{i,\tau}'\bar{\mathbf{F}}'\boldsymbol{\epsilon}_j'\right)^2 = o_p(1);$
4. $S_4 = \frac{1}{\sqrt{N(N-1)}}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{\sigma_i\sigma_j}{\sqrt{T}}\boldsymbol{\epsilon}_i'\boldsymbol{\epsilon}_j\frac{1}{T^{1/2+2\alpha_T}N^{2\alpha_N}}\boldsymbol{\ell}_{i,\tau}'\bar{\mathbf{F}}'\bar{\mathbf{F}}\boldsymbol{\ell}_{j,\tau} = o_p(1);$
5. $S_5 = \frac{1}{\sqrt{N(N-1)}}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{\sigma_i\sigma_j}{\sqrt{T}}\boldsymbol{\epsilon}_i'\boldsymbol{\epsilon}_j\frac{\sigma_i}{T^{1/2+\alpha_T}N^{\alpha_N}}\boldsymbol{\epsilon}_i'\bar{\mathbf{F}}\boldsymbol{\ell}_{j,\tau} = o_p(1);$
6. $S_6 = \frac{1}{\sqrt{N(N-1)}}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{\sigma_i\sigma_j}{\sqrt{T}}\boldsymbol{\epsilon}_i'\boldsymbol{\epsilon}_j\frac{\sigma_j}{T^{1/2+\alpha_T}N^{\alpha_N}}\boldsymbol{\ell}_{i,\tau}'\bar{\mathbf{F}}'\boldsymbol{\epsilon}_j' = o_p(1);$

7. $S_7 = \frac{1}{\sqrt{N(N-1)}} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{1}{T^{1/2+2\alpha_T} N^{2\alpha_N}} \boldsymbol{\ell}_{i,\tau}' \bar{\mathbf{F}}' \bar{\mathbf{F}} \boldsymbol{\ell}_{j,\tau} \frac{\sigma_i}{T^{1/2+\alpha_T} N^{\alpha_N}} \boldsymbol{\epsilon}_i' \bar{\mathbf{F}} \boldsymbol{\ell}_{j,\tau} = o_p(1)$;

8. $S_8 = \frac{1}{\sqrt{N(N-1)}} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{1}{T^{1/2+2\alpha_T} N^{2\alpha_N}} \boldsymbol{\ell}_{i,\tau}' \bar{\mathbf{F}}' \bar{\mathbf{F}} \boldsymbol{\ell}_{j,\tau} \frac{\sigma_j}{T^{1/2+\alpha_T} N^{\alpha_N}} \boldsymbol{\ell}_{i,\tau}' \bar{\mathbf{F}}' \boldsymbol{\epsilon}_j' = o_p(1)$;

9. $S_9 = \frac{1}{\sqrt{N(N-1)}} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{\sigma_i}{T^{1/2+\alpha_T} N^{\alpha_N}} \boldsymbol{\epsilon}_i' \bar{\mathbf{F}} \boldsymbol{\ell}_{j,\tau} \frac{\sigma_j}{T^{1/2+\alpha_T} N^{\alpha_N}} \boldsymbol{\ell}_{i,\tau}' \bar{\mathbf{F}}' \boldsymbol{\epsilon}_j' = o_p(1)$.

Appendix B - Proofs of main results

Proof of Lemma 1

We begin with $Q_1 = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{T}{\sigma_i^2 \sigma_j^2} \left(\frac{1}{T} \left(\hat{\boldsymbol{\beta}}_{i,\tau} - \boldsymbol{\beta}_i \right)' \ddot{\mathbf{X}}_i' \ddot{\mathbf{u}}_j \right)^2$ for which we have

$$
\begin{aligned}
Q_1 &= \frac{1}{T} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{1}{\sigma_i^2 \sigma_j^2} \left(\sqrt{T} \left(\hat{\boldsymbol{\beta}}_{i,\tau} - \boldsymbol{\beta}_i \right)' \frac{1}{\sqrt{T}} \ddot{\mathbf{X}}_i' \ddot{\mathbf{u}}_j \right)^2 \\
&= \frac{1}{T} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{1}{\sigma_i^2 \sigma_j^2} \operatorname{E} \left(\sqrt{T} \left(\hat{\boldsymbol{\beta}}_{i,\tau} - \boldsymbol{\beta}_i \right)' \frac{1}{\sqrt{T}} \ddot{\mathbf{X}}_i' \ddot{\mathbf{u}}_j \right)^2 \\
&\quad + \frac{1}{T} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left[\frac{1}{\sigma_i^2 \sigma_j^2} \left(\sqrt{T} \left(\hat{\boldsymbol{\beta}}_{i,\tau} - \boldsymbol{\beta}_i \right)' \frac{1}{\sqrt{T}} \ddot{\mathbf{X}}_i' \ddot{\mathbf{u}}_j \right)^2 \right. \\
&\qquad \left. - \frac{1}{\sigma_i^2 \sigma_j^2} \operatorname{E} \left(\sqrt{T} \left(\hat{\boldsymbol{\beta}}_{i,\tau} - \boldsymbol{\beta}_i \right)' \frac{1}{\sqrt{T}} \ddot{\mathbf{X}}_i' \ddot{\mathbf{u}}_j \right)^2 \right] \\
&= Q_{1,1} + Q_{1,2}.
\end{aligned}
$$

Under the null of no cross-sectional dependence and with Assumptions 3 and 5, we note that R_{NT} does not play any role asymptotically in either summand, so, with a mild abuse of procedure, we set it to 0 and obtain

$$
\begin{aligned}
Q_{1,1} &= \frac{1}{T} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{1}{\sigma_i^2} \operatorname{E} \left(\sqrt{T} \left(\hat{\boldsymbol{\beta}}_{i,\tau} - \boldsymbol{\beta}_i \right)' \right)^2 \boldsymbol{\Sigma}_i + o_p \left(\frac{N^2}{T} \right) \\
&= \frac{N(N-1)\tau(1-\tau)}{2T f(q_\tau)^2} + o_p \left(\frac{N^2}{T} \right).
\end{aligned}
$$

For $Q_{1,2}$ we have, by means of Assumption 5 and no dependence under the null, that

$$
\begin{aligned}
\frac{T}{N^2} Q_{1,2} &= \frac{T^2}{N^2} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left[\frac{1}{\sigma_i^2 \sigma_j^2} \left(\frac{1}{T} \left(\hat{\boldsymbol{\beta}}_{i,\tau} - \boldsymbol{\beta}_{i,\tau} \right)' \ddot{\mathbf{X}}_i' \ddot{\mathbf{u}}_j \right)^2 \right. \\
&\qquad \left. - \frac{1}{\sigma_i^2 \sigma_j^2} \operatorname{Var} \left(\frac{1}{T} \left(\hat{\boldsymbol{\beta}}_{i,\tau} - \boldsymbol{\beta}_{i,\tau} \right)' \ddot{\mathbf{X}}_i' \ddot{\mathbf{u}}_j \right) \right] \\
&= C \frac{T}{N^{2-\delta}} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left(\left(\frac{1}{T} \mathbf{1}' \ddot{\mathbf{X}}_i' \ddot{\mathbf{u}}_j \right)^2 - \operatorname{E} \left(\frac{1}{T} \mathbf{1}' \ddot{\mathbf{X}}_i' \ddot{\mathbf{u}}_j \right)^2 \right) \\
&= O_p \left(N^{-\delta} \right),
\end{aligned}
$$

since $\mathrm{E}\left|\left(\frac{1}{T}\mathbf{1}'\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j\right)^2 - \mathrm{E}\left(\frac{1}{T}\mathbf{1}'\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j\right)^2\right| \leq \frac{C}{T}$.

Using similar arguments for Q_2 we obtain

$$Q_1 + Q_2 = \frac{N(N-1)}{T}\frac{\tau(1-\tau)}{f(q_\tau)^2} + O_p(1),$$

which after scaling by $\frac{1}{\sqrt{N(N-1)}}$, constitutes the second correction term suggested in (5).

We now turn to Q_3,

$$\begin{aligned}\frac{1}{N}Q_3 &= \frac{1}{N}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} T\frac{\left(\frac{1}{T}\left(\hat{\beta}_{i,\tau}-\beta_i\right)'\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\left(\hat{\beta}_{i,\tau}-\beta_i\right)\right)^2}{\sigma_i^2\sigma_j^2} \\ &= O_p\left(\frac{N^{2\delta}}{T^2}\right)\frac{1}{N}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{\left(\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\right)^2}{T} = O_p\left(\frac{N^{2\delta+1}}{T}\right),\end{aligned}$$

since

$$\begin{aligned}\frac{1}{NT}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\mathrm{E}\left(\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\right)^2 &= \frac{1}{NT}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\mathrm{E}\left(\sum_{t=1}^{T}\sum_{s=1}^{T}\ddot{x}_{i,t}\ddot{x}_{j,t}\ddot{x}_{i,s}\ddot{x}_{j,s}\right) \\ &= O(NT),\end{aligned}$$

and, thanks to assumption 3,

$$\begin{aligned}\frac{1}{N^2T^2}\mathrm{E}\left(\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\right)^2\right)^2 &= \frac{1}{N^2T^2}\mathrm{E}\left(\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\sum_{k=1}^{N-1}\sum_{l=k+1}^{N}\left(\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\right)^2\left(\ddot{\mathbf{X}}_k'\ddot{\mathbf{X}}_l\right)^2\right) \\ &= \frac{1}{N^2T^2}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\sum_{k=1}^{N-1}\sum_{l=k+1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{t'=1}^{T}\sum_{s'=1}^{T}\mathrm{E}\left(\ddot{x}_{it}\ddot{x}_{jt}\ddot{x}_{is}\ddot{x}_{js}\ddot{x}_{it'}\ddot{x}_{jt'}\ddot{x}_{is'}\ddot{x}_{js'}\right) \\ &= O\left(N^2T^2\right).\end{aligned}$$

For Q_4 we have

$$\begin{aligned}\frac{1}{N}Q_4 &= -\frac{2}{N}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\left(\hat{\beta}_{i,\tau}-\beta_i\right)'\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j}{T\sigma_i^2\sigma_j^2} \\ &= O_p\left(\frac{N^{\delta/2}}{T^{1/2}}\right)\frac{1}{NT}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j}{\sigma_i^2\sigma_j^2},\end{aligned}$$

Since

$$\frac{1}{NT}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j = \frac{1}{NT}\sum_{j=2}^{N}\sum_{i=1}^{j-1}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j$$

we have, thanks to independence of $\{\boldsymbol{u}_i\}$, that

$$\frac{1}{N^2T^2}\operatorname{Var}\left(\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j\right)=\frac{1}{N^2T^2}\sum_{j=2}^{N}\sum_{i=1}^{j-1}\mathrm{E}\left(\left(\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j\right)^2|\ddot{\mathbf{X}}_i,\,\forall i\right).$$

But $\mathrm{E}\left(\left(\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j\right)^2|x_i,\,\forall i\right)=O\left(T^2\right)$, since

$$\begin{aligned}
\mathrm{E}\left(\left(\sum_{t=1}^{T}\ddot{u}_{i,t}\ddot{u}_{j,t}\right)^2\left(\sum_{t=1}^{T}\ddot{x}_{i,t}\ddot{u}_{i,t}\right)^2\right) &= \sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{t'=1}^{T}\sum_{s'=1}^{T}\mathrm{E}\left(\ddot{u}_{it}\ddot{u}_{jt}\ddot{u}_{is}\ddot{u}_{js}\ddot{x}_{it'}\ddot{u}_{it'}\ddot{x}_{is'}\ddot{u}_{is'}\right)\\
&= \sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{t'=1}^{T}\mathrm{E}\left(\ddot{u}_{it}\ddot{u}_{jt}\ddot{u}_{is}\ddot{u}_{js}\ddot{x}_{it'}^2\ddot{u}_{it'}^2\right)\\
&\quad+\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{t'=1}^{T}\sum_{s'\neq t',s'=1}^{T}\mathrm{E}\left(\ddot{u}_{it}\ddot{u}_{jt}\ddot{u}_{is}\ddot{u}_{js}\ddot{x}_{it'}\ddot{u}_{it'}\ddot{x}_{is'}\ddot{u}_{is'}\right)\\
&= \sum_{t=1}^{T}\sum_{t'=1}^{T}\mathrm{E}\left(\ddot{u}_{it}^2\ddot{u}_{jt}^2\ddot{x}_{it'}^2\ddot{u}_{it'}^2\right)+\sum_{t=1}^{T}\sum_{s\neq t,s=1}^{T}\sum_{t'=1}^{T}\mathrm{E}\left(\ddot{u}_{it}\ddot{u}_{jt}\ddot{u}_{is}\ddot{u}_{js}\ddot{x}_{it'}^2\ddot{u}_{it'}^2\right)\\
&\quad+\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{t'=1}^{T}\sum_{s'\neq t',s'=1}^{T}\mathrm{E}\left(\ddot{u}_{it}\ddot{u}_{jt}\ddot{u}_{is}\ddot{u}_{js}\ddot{x}_{it'}\ddot{u}_{it'}\ddot{x}_{is'}\ddot{u}_{is'}\right)
\end{aligned}$$

which is $O\left(T^2\right)$, and therefore $Q_4=O_p\left(\frac{N^{1+\delta/2}}{T^{1/2}}\right)$.

Q_5 is similar to Q_4.

For Q_6 we have

$$\begin{aligned}
\frac{1}{N}Q_6 &= \frac{2}{N}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\left(\hat{\beta}_{i,\tau}-\beta_i\right)'\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\left(\hat{\beta}_{i,\tau}-\beta_i\right)}{T\sigma_i^2\sigma_j^2}\\
&= O_p\left(\frac{N^\delta}{T}\right)\frac{1}{NT}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\\
&= O_p\left(\frac{N^\delta}{T}\right)\frac{1}{NT}\sum_{j=2}^{N}\sum_{i=1}^{j-1}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j
\end{aligned}$$

where again using the same arguments as for Q_4 we obtain $Q_6=O_p\left(\frac{N^{\delta+1}}{T}\right)$.

Q_7 is similar to Q_6.

For Q_8 we have

$$\begin{aligned}
\frac{1}{N}Q_8 &= -2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{\left(\hat{\beta}_{i,\tau}-\beta_i\right)'\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j\left(\hat{\beta}_{i,\tau}-\beta_i\right)'\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\left(\hat{\beta}_{i,\tau}-\beta_i\right)}{T\sigma_i^2\sigma_j^2}\\
&= O_p\left(\frac{N^{3\delta/2}}{T^{3/2}}\right)\frac{1}{NT}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j.
\end{aligned}$$

The second moment of $\frac{1}{NT}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j$ is $O(T)$, since

$$\begin{aligned}
\mathrm{E}\left(\frac{1}{NT}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\right)^2 &= \frac{1}{N^2T^2}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\sum_{k=1}^{N-1}\sum_{l=k+1}^{N}\mathrm{E}\left(\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\ddot{\mathbf{X}}_k'\ddot{\boldsymbol{u}}_l\ddot{\mathbf{X}}_k'\ddot{\mathbf{X}}_l\right)\\
&= \frac{1}{N^2T^2}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\mathrm{E}\left(\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\right)\\
&= \frac{1}{N^2T^2}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\mathrm{E}\left(\sum_{t=1}^{T}\sum_{s=1}^{T}\ddot{x}_{i,t}\ddot{u}_{j,t}\ddot{x}_{i,s}\ddot{x}_{j,s}\right)^2\\
&= \frac{1}{N^2T^2}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\mathrm{E}\left(\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{t'=1}^{T}\sum_{s'=1}^{T}\ddot{x}_{it}\ddot{u}_{jt}\ddot{x}_{is}\ddot{x}_{js}\ddot{x}_{it'}\ddot{u}_{jt'}\ddot{x}_{is'}\ddot{x}_{js'}\right)\\
&= \frac{1}{N^2T^2}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{s'=1}^{T}\mathrm{E}\left(\ddot{x}_{it}\ddot{u}_{jt}^2\ddot{x}_{is}\ddot{x}_{js}\ddot{x}_{it'}\ddot{x}_{is'}\ddot{x}_{js'}\right)\\
&= \frac{C}{N^2T^2}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{s'=1}^{T}\mathrm{E}\left(\ddot{x}_{it}\ddot{x}_{is}\ddot{x}_{js}\ddot{x}_{it'}\ddot{x}_{is'}\ddot{x}_{js'}\right)\\
&= O\left(T\right),
\end{aligned}$$

for some constant C. Hence $Q_8 = O_p\left(\frac{N^{3\delta/2+1}}{T}\right)$.

Q_9 behaves similarly to Q_8.

Now we may analyse Q_{10}.

$$Q_{10} = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} T\frac{\widehat{\sigma}_{ij}^2\left(1-\frac{\widehat{\sigma}_i^2\widehat{\sigma}_j^2}{\sigma_i^2\sigma_j^2}\right)}{\widehat{\sigma}_i^2\widehat{\sigma}_j^2}.$$

Note that

$$\begin{aligned}
\widehat{\sigma}_i^2 &= \frac{1}{T}\left(\ddot{\boldsymbol{u}}_i-\ddot{\mathbf{X}}_i\left(\hat{\beta}_{i,\tau}-\beta_i\right)\right)'\left(\ddot{\boldsymbol{u}}_i-\ddot{\mathbf{X}}_i\left(\hat{\beta}_{i,\tau}-\beta_i\right)\right)\\
&= \frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_i-\frac{1}{T}\left(\hat{\beta}_{i,\tau}-\beta_i\right)'\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_i-\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\mathbf{X}}_i\left(\hat{\beta}_{i,\tau}-\beta_i\right)+\frac{1}{T}\left(\hat{\beta}_{i,\tau}-\beta_i\right)'\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_i\left(\hat{\beta}_{i,\tau}-\beta_i\right)\\
&= \frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_i+B_i
\end{aligned}$$

where B_i is defined implicitly. We therefore may write

$$1-\frac{\widehat{\sigma}_i^2\widehat{\sigma}_j^2}{\sigma_i^2\sigma_j^2} = 1-\frac{\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_i\frac{1}{T}\ddot{\boldsymbol{u}}_j'\ddot{\boldsymbol{u}}_j}{\sigma_i^2\sigma_j^2}-\frac{B_j\left(\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_i\right)+B_i\left(\frac{1}{T}\ddot{\boldsymbol{u}}_j'\ddot{\boldsymbol{u}}_j+B_j\right)}{\sigma_i^2\sigma_j^2}.$$

Plugging the latter into the expression for Q_{10} we obtain

$$Q_{10} = -\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} T\frac{\widehat{\sigma}_{ij}^2}{\widehat{\sigma}_i^2\widehat{\sigma}_j^2}\left(1-\frac{\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_i\frac{1}{T}\ddot{\boldsymbol{u}}_j'\ddot{\boldsymbol{u}}_j}{\sigma_i^2\sigma_j^2}-\frac{B_j\left(\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_i\right)+B_i\left(\frac{1}{T}\ddot{\boldsymbol{u}}_j'\ddot{\boldsymbol{u}}_j+B_j\right)}{\sigma_i^2\sigma_j^2}\right).$$

which together with $\frac{\widehat{\sigma}_{ij}^2}{\widehat{\sigma}_i^2\widehat{\sigma}_j^2} = \frac{\widehat{\sigma}_{ij}^2}{\sigma_i^2\sigma_j^2} + \frac{\widehat{\sigma}_{ij}^2}{\widehat{\sigma}_i^2\widehat{\sigma}_j^2} - \frac{\widehat{\sigma}_{ij}^2}{\sigma_i^2\sigma_j^2}$ can be written as

$$
\begin{aligned}
\frac{1}{N}Q_{10} &= -\frac{1}{N}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} T\frac{\widehat{\sigma}_{ij}^2}{\sigma_i^2\sigma_j^2}\left(1 - \frac{\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_i\frac{1}{T}\ddot{\boldsymbol{u}}_j'\ddot{\boldsymbol{u}}_j}{\sigma_i^2\sigma_j^2} - \frac{B_j\left(\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_i\right) + B_i\left(\frac{1}{T}\ddot{\boldsymbol{u}}_j'\ddot{\boldsymbol{u}}_j + B_j\right)}{\sigma_i^2\sigma_j^2}\right) \\
&\quad -\frac{1}{N}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} T\frac{\widehat{\sigma}_{ij}^2}{\sigma_i^2\sigma_j^2}\left(\frac{1}{\widehat{\sigma}_i^2\widehat{\sigma}_j^2} - \frac{1}{\sigma_i^2\sigma_j^2}\right)\left(1 - \frac{\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_i\frac{1}{T}\ddot{\boldsymbol{u}}_j'\ddot{\boldsymbol{u}}_j}{\sigma_i^2\sigma_j^2} - \frac{B_j\left(\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_i\right) + B_i\left(\frac{1}{T}\ddot{\boldsymbol{u}}_j'\ddot{\boldsymbol{u}}_j + B_j\right)}{\sigma_i^2\sigma_j^2}\right) \\
&= -P_1 - P_2
\end{aligned}
$$

Since P_2 involves $\frac{1}{\widehat{\sigma}_i^2\widehat{\sigma}_j^2} - \frac{1}{\sigma_i^2\sigma_j^2} = O_p\left(T^{-1/2}\right)$ (uniformly), it is clearly dominated by P_1, hence we only have to show that P_1 vanishes at a required rate. Write

$$
\begin{aligned}
P_1 &= \frac{1}{N}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} T\frac{\widehat{\sigma}_{ij}^2}{\sigma_i^2\sigma_j^2}\left(1 - \frac{\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_i\frac{1}{T}\ddot{\boldsymbol{u}}_j'\ddot{\boldsymbol{u}}_j}{\sigma_i^2\sigma_j^2} - \frac{B_j\left(\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_i\right) + B_i\left(\frac{1}{T}\ddot{\boldsymbol{u}}_j'\ddot{\boldsymbol{u}}_j + B_j\right)}{\sigma_i^2\sigma_j^2}\right) \\
&= \frac{1}{N}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} T\frac{\widehat{\sigma}_{ij}^2}{\sigma_i^2\sigma_j^2}\left(1 - \frac{\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_i\frac{1}{T}\ddot{\boldsymbol{u}}_j'\ddot{\boldsymbol{u}}_j}{\sigma_i^2\sigma_j^2}\right) \\
&- \frac{1}{N}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} T\frac{\widehat{\sigma}_{ij}^2}{\sigma_i^2\sigma_j^2}\left(\frac{B_j\left(\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_i\right) + B_i\left(\frac{1}{T}\ddot{\boldsymbol{u}}_j'\ddot{\boldsymbol{u}}_j + B_j\right)}{\sigma_i^2\sigma_j^2}\right)
\end{aligned}
$$

First we note that $1 - \frac{\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_i\frac{1}{T}\ddot{\boldsymbol{u}}_j'\ddot{\boldsymbol{u}}_j}{\sigma_i^2\sigma_j^2}$ is $O_p\left(T^{-1}\right)$ uniformly under the null, second, using Assumption 5, we observe that $B_j\left(\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_i\right) + B_i\left(\frac{1}{T}\ddot{\boldsymbol{u}}_j'\ddot{\boldsymbol{u}}_j + B_j\right)$ is dominated by B_iB_j which turns out to be $O_p\left(\frac{N^\delta}{T}\right)$ uniformly and third $\mathrm{Var}\left(\sqrt{T}\frac{\widehat{\sigma}_{ij}}{\sigma_i\sigma_j}\right)^2 = C$ for some constant C. Therefore $P_1 = O_p\left(\frac{N^\delta}{T}\right)$ which in turn implies $Q_{10} = O_p\left(\frac{N^{1+\delta}}{T}\right)$.

Proof of Lemma 2

For item 1, we have

$$
\begin{aligned}
S_1 &= \frac{1}{N\sqrt{N(N-1)}}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\boldsymbol{\ell}_{i,\tau}'\left(\frac{1}{T}\tilde{\mathbf{F}}'\tilde{\mathbf{F}} - \boldsymbol{\Sigma}_f + \boldsymbol{\Sigma}_f\right)\boldsymbol{\ell}_{j,\tau}\right)^2 \\
&= \frac{1}{N\sqrt{N(N-1)}}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\left(\boldsymbol{\ell}_{i,\tau}'\boldsymbol{\Sigma}_f\boldsymbol{\ell}_{j,\tau}\right)^2 + \left(\boldsymbol{\ell}_{i,\tau}'\left(\frac{1}{T}\tilde{\mathbf{F}}'\tilde{\mathbf{F}} - \boldsymbol{\Sigma}_f\right)\boldsymbol{\ell}_{j,\tau}\right)^2 + 2\left(\boldsymbol{\ell}_{i,\tau}'\boldsymbol{\Sigma}_f\boldsymbol{\ell}_{j,\tau}\right)\left(\boldsymbol{\ell}_{i,\tau}'\left(\frac{1}{T}\tilde{\mathbf{F}}'\tilde{\mathbf{F}} - \boldsymbol{\Sigma}_f\right)\boldsymbol{\ell}_{j,\tau}\right.\right.
\end{aligned}
$$

where the summation over the first term is by assumption $c_\tau^2 + o(1)$ and the summation over the second and third terms is $o_p(1)$ with stationarity and bounded 4th moment assumption on $\boldsymbol{f}_t$.

For part 2 we have

$$
S_2 = \frac{T}{T^{1/2}N^{1/2}\sqrt{N(N-1)}}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{\sigma_i}{T}\tilde{\boldsymbol{\epsilon}}_i'\tilde{\mathbf{F}}\boldsymbol{\ell}_{j,\tau}\right)^2.
$$

Due to the zero-mean and independence of $\tilde{\boldsymbol{\epsilon}}_i'$ and $\tilde{\mathbf{F}}$ we have that $\mathrm{E}\left(\frac{\sigma_i}{T}\tilde{\boldsymbol{\epsilon}}_i'\tilde{\mathbf{F}}\boldsymbol{\ell}_{j,\tau}\right)^2 = \frac{C}{T}$ for some constant

C which in turn implies that $S_2 = O_p\left(\frac{N^{1/2}}{T^{1/2}}\right) = o_p(1)$.

S_3 is dealt with in a manner similar to S_2.

For S_4, we have

$$S_4 = \frac{1}{\sqrt{N(N-1)}} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{\sigma_i \sigma_j}{\sqrt{T}} \tilde{\boldsymbol{\epsilon}}_i' \tilde{\boldsymbol{\epsilon}}_j \frac{1}{TN^{1/2}} \boldsymbol{\ell}_{i,\tau}' \tilde{\mathbf{F}}' \tilde{\mathbf{F}} \boldsymbol{\ell}_{j,\tau}$$

where, with the independence of $\tilde{\boldsymbol{\epsilon}}_i' \tilde{\boldsymbol{\epsilon}}_j$ and $\boldsymbol{\ell}_{i,\tau}' \tilde{\mathbf{F}}' \tilde{\mathbf{F}} \boldsymbol{\ell}_{j,\tau}$, the fact that $\mathrm{E}\left(\tilde{\boldsymbol{\epsilon}}_i' \tilde{\boldsymbol{\epsilon}}_j\right) = 0$, and the stationarity and bounded 4th moment assumption on $\boldsymbol{f}_t$, we obtain $\mathrm{E}\left(\left|\frac{1}{\sqrt{T}} \tilde{\boldsymbol{\epsilon}}_i' \tilde{\boldsymbol{\epsilon}}_j \frac{1}{T} \boldsymbol{\ell}_{i,\tau}' \tilde{\mathbf{F}}' \tilde{\mathbf{F}} \boldsymbol{\ell}_{j,\tau}\right|\right) \leq \frac{C}{T}$ and hence $S_4 = O_p\left(\frac{N^{1/2}}{T}\right) = o_p(1)$.

Using similar arguments, S_5 up to S_9 are then all shown to be $o_p(1)$.

Proof of Proposition 1

We have that

$$\hat{u}_{i,t,\tau} = u_{i,t} - (\hat{\alpha}_{i,\tau} - \alpha_{i,\tau}) - \left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right)' \boldsymbol{x}_{i,t}$$

such that, with $\bar{\cdot}$ denoting the unit specific sample mean, we have $\bar{\hat{u}}_{i,\tau} = \bar{u}_i - (\hat{\alpha}_{i,\tau} - \alpha_{i,\tau}) - \left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right)' \bar{\boldsymbol{x}}_i$ and therefore

$$\hat{\boldsymbol{u}}_{i,\tau} - \bar{\hat{u}}_{i,\tau} \boldsymbol{\iota} = \ddot{\boldsymbol{u}}_i - \ddot{\mathbf{X}}_i \left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right).$$

Then,

$$\begin{aligned}
\widehat{\sigma}_{ij}^2 &= \left(\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j - \frac{1}{T}\left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right)'\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j - \frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right) + \frac{1}{T}\left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right)'\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right)\right)^2 \\
&= \left(\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\right)^2 + \left(\frac{1}{T}\left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right)'\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j\right)^2 + \left(\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right)\right)^2 + \left(\frac{1}{T}\left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right)'\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right)\right)^2 \\
&\quad - \frac{2}{T^2}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right)'\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j - \frac{2}{T^2}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\ddot{\boldsymbol{u}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right) + \frac{2}{T^2}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right)'\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right) \\
&\quad + \frac{2}{T^2}\left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right)'\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j\ddot{\boldsymbol{u}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right) - \frac{2}{T^2}\left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right)'\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j\left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right)'\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right) \\
&\quad - \frac{2}{T^2}\ddot{\boldsymbol{u}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right)\left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right)'\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau\right) \\
&= \left(\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\right)^2 + \sum_{k=1}^{9} A_{k,ij}.
\end{aligned}$$

with $A_{k,ij}$ defined implicitly. Write now

$$\begin{aligned}
\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{T\widehat{\sigma}_{ij}^2}{\widehat{\sigma}_i^2\widehat{\sigma}_j^2} - 1\right) &= \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{T\widehat{\sigma}_{ij}^2}{\sigma_i^2\sigma_j^2} - 1 + \frac{T\widehat{\sigma}_{ij}^2}{\sigma_i^2\sigma_j^2}\left(\frac{\sigma_i^2\sigma_j^2}{\widehat{\sigma}_i^2\widehat{\sigma}_j^2} - 1\right)\right) \\
&= \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{\frac{1}{T}\left(\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\right)^2}{\sigma_i^2\sigma_j^2} - 1\right) + \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} T\frac{\sum_{k=1}^{9} A_{k,ij}}{\sigma_i^2\sigma_j^2} + \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} T\frac{\widehat{\sigma}_{ij}^2\left(1 - \frac{\widehat{\sigma}_i^2\widehat{\sigma}_j^2}{\sigma_i^2\sigma_j^2}\right)}{\widehat{\sigma}_i^2\widehat{\sigma}_j^2}
\end{aligned}$$

$$= \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{T\left(\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\right)^2}{\sigma_i^2\sigma_j^2}-1\right)+\sum_{k=1}^{10}Q_k,$$

where the rest terms Q_k, $k=1,\ldots,10$, are shown in Lemma 1 to be $o_p(N)$ under the weaker conditions of individual-unit estimation. Therefore

$$\frac{1}{\sqrt{N(N-1)}}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{T\widehat{\sigma}_{ij}^2}{\widehat{\sigma}_i^2\widehat{\sigma}_j^2}-1\right)=\frac{1}{\sqrt{N(N-1)}}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{T\left(\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\right)^2}{\sigma_i^2\sigma_j^2}-1\right)+o_p(1).$$

Let now $\mu_i=\mathrm{E}(u_{i,t})=\sigma_i\,\mathrm{E}(\epsilon_{i,t})+\boldsymbol{\lambda}_i'\,\mathrm{E}(\boldsymbol{f}_t)$ and write

$$\begin{aligned}
T\left(\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\right)^2 &= \left(\frac{1}{\sqrt{T}}(\boldsymbol{u}_i-\boldsymbol{\iota}\mu_i)'(\boldsymbol{u}_j-\boldsymbol{\iota}\mu_j)-\sqrt{T}(\bar{u}_i-\mu_i)(\bar{u}_j-\mu_j)\right)^2\\
&= \left(\frac{1}{\sqrt{T}}(\boldsymbol{u}_i-\boldsymbol{\iota}\mu_i)'(\boldsymbol{u}_j-\boldsymbol{\iota}\mu_j)\right)^2-2\frac{1}{\sqrt{T}}(\boldsymbol{u}_i-\boldsymbol{\iota}\mu_i)'(\boldsymbol{u}_j-\boldsymbol{\iota}\mu_j)(\bar{u}_i-\mu_i)\sqrt{T}(\bar{u}_j-\mu_j)\\
&\quad +T(\bar{u}_i-\mu_i)^2(\bar{u}_j-\mu_j)^2.
\end{aligned}$$

We show below that $\frac{1}{\sqrt{N(N-1)}}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{\left(\frac{1}{\sqrt{T}}(\boldsymbol{u}_i-\boldsymbol{\iota}\mu_i)'(\boldsymbol{u}_j-\boldsymbol{\iota}\mu_j)\right)^2}{\sigma_i^2\sigma_j^2}-1\right)\xrightarrow{d}\mathcal{N}\left(c_\tau^2,1\right)$ under our rate restrictions, so the result follows if the second and third terms on the r.h.s. vanish. We examine the vanishing terms in turn.

With $\tilde{\mathbf{F}}$ stacking $\tilde{\boldsymbol{f}}_t'=\boldsymbol{f}_t'-\mathrm{E}\left(\boldsymbol{f}_t'\right)$ and $\tilde{\boldsymbol{\epsilon}}_i$ stacking $\epsilon_{i,t}-\mathrm{E}(\epsilon_{i,t})$, we have

$$\boldsymbol{u}_i-\boldsymbol{\iota}\mu_i=\sigma_i\tilde{\boldsymbol{\epsilon}}_i+\boldsymbol{\lambda}_i'\tilde{\mathbf{F}}=\sigma_i\tilde{\boldsymbol{\epsilon}}_i+\frac{1}{T^{1/4}N^{1/4}}\boldsymbol{\ell}_{i,\tau}'\tilde{\mathbf{F}},$$

such that

$$\begin{aligned}
(\boldsymbol{u}_i-\boldsymbol{\iota}\mu_i)'(\boldsymbol{u}_j-\boldsymbol{\iota}\mu_j)(\bar{u}_i-\mu_i)(\bar{u}_j-\mu_j) &= \left(\sigma_i\tilde{\boldsymbol{\epsilon}}_i+\frac{1}{T^{1/4}N^{1/4}}\boldsymbol{\ell}_{i,\tau}'\tilde{\mathbf{F}}\right)'\left(\sigma_j\tilde{\boldsymbol{\epsilon}}_j+\frac{1}{T^{1/4}N^{1/4}}\boldsymbol{\ell}_{j,\tau}'\tilde{\mathbf{F}}\right)\\
&\quad\times\left(\sigma_i\bar{\tilde{\epsilon}}_i+\frac{1}{T^{1/4}N^{1/4}}\boldsymbol{\ell}_{i,\tau}'\bar{\tilde{\boldsymbol{f}}}\right)\left(\sigma_j\bar{\tilde{\epsilon}}_j+\frac{1}{T^{1/4}N^{1/4}}\boldsymbol{\ell}_{j,\tau}'\bar{\tilde{\boldsymbol{f}}}\right)
\end{aligned}$$

where it is tedious, yet straightforward to show that this is

$$(\sigma_i\tilde{\boldsymbol{\epsilon}}_i)'(\sigma_j\tilde{\boldsymbol{\epsilon}}_j)(\sigma_i\bar{\tilde{\epsilon}}_i)(\sigma_j\bar{\tilde{\epsilon}}_j)+\mathit{negligible}:=\frac{1}{\sqrt{N(N-1)}}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}a_{ij}b_ib_j+\mathit{negligible}.$$

It is then not difficult to show that the expectation of the leading term on the r.h.s. is $O_p\left(\frac{N}{T^2}\right)$. Moreover, since b_i and b_j are zero-mean independent quantities, and also independent of a_{kl} for any $i\neq k, j\neq l$ it can be seen that the products $a_{ij}b_ib_j$ are pairwise uncorrelated and, given the moment requirements on $\tilde{\epsilon}_{i,t}$, also have finite variance. Therefore,

$$\mathrm{Var}\left(\frac{1}{\sqrt{N(N-1)}}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}a_{ij}b_ib_j\right)=\frac{1}{N(N-1)}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\mathrm{Var}\left((\sigma_i\tilde{\boldsymbol{\epsilon}}_i)'(\sigma_j\tilde{\boldsymbol{\epsilon}}_j)(\sigma_i\bar{\tilde{\epsilon}}_i)(\sigma_j\bar{\tilde{\epsilon}}_j)\right),$$

where the individual variances on the r.h.s. are in turn $O\left(T^{-1}\right)$. Therefore, Chebyshev's inequality ultimately leads to

$$\frac{1}{\sqrt{N\left(N-1\right)}}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}a_{ij}b_ib_j=O_p\left(\max\left\{\frac{1}{\sqrt{T}};\frac{N}{T^2}\right\}\right)=o_p(1).$$

To complete the analysis, note that the leading term of $T\left(\bar{u}_i-\mu_i\right)^2\left(\bar{u}_j-\mu_j\right)^2$ is $T\sigma_i^2\sigma_j^2\left(\bar{\tilde{\epsilon}}_i\right)^2\left(\bar{\tilde{\epsilon}}_j\right)^2$, for which

$$0\leq \mathrm{E}\left(T\sigma_i^2\sigma_j^2\left(\bar{\tilde{\epsilon}}_i\right)^2\left(\bar{\tilde{\epsilon}}_j\right)^2\right)=\frac{\sigma_i^2\sigma_j^2}{T},$$

so, thanks to Markov's inequality,

$$\frac{1}{\sqrt{N(N-1)}}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{T\sigma_i^2\sigma_j^2\left(\bar{\tilde{\epsilon}}_i\right)^2\left(\bar{\tilde{\epsilon}}_j\right)^2}{\sigma_i^2\sigma_j^2}=O_p\left(\frac{N}{T}\right)=o_p(1)$$

under our rate conditions.

Importantly, the expectation of this term is given by $\frac{\sqrt{N(N-1)}}{2T}$, which justifies the first component of the finite-sample correction proposed in (5). The second component of the correction is obtained from Lemma 1, stemming from the leading term of Q_1+Q_2.

To conclude, we have

$$\frac{1}{\sqrt{T}}\left(\boldsymbol{u}_i-\boldsymbol{\iota}\mu_i\right)'\left(\boldsymbol{u}_j-\boldsymbol{\iota}\mu_j\right)=\frac{\sigma_i\sigma_j}{\sqrt{T}}\tilde{\boldsymbol{\epsilon}}_i'\tilde{\boldsymbol{\epsilon}}_j+\frac{1}{TN^{1/2}}\boldsymbol{\ell}_{i,\tau}'\tilde{\mathbf{F}}'\tilde{\mathbf{F}}\boldsymbol{\ell}_{j,\tau}+\frac{\sigma_i}{T^{3/4}N^{1/4}}\tilde{\boldsymbol{\epsilon}}_i'\tilde{\mathbf{F}}\boldsymbol{\ell}_{j,\tau}+\frac{\sigma_j}{T^{3/4}N^{1/4}}\boldsymbol{\ell}_{i,\tau}'\tilde{\mathbf{F}}'\tilde{\boldsymbol{\epsilon}}_j'$$

Upon squaring the r.h.s., Lemma 2 then indicates that

$$\begin{aligned}&\frac{1}{\sqrt{N\left(N-1\right)}}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{\left(\frac{1}{\sqrt{T}}\left(\boldsymbol{u}_i-\boldsymbol{\iota}\mu_i\right)'\left(\boldsymbol{u}_j-\boldsymbol{\iota}\mu_j\right)\right)^2}{\sigma_i^2\sigma_j^2}-1\right)\\&=\frac{1}{\sqrt{N\left(N-1\right)}}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\left(\frac{1}{\sqrt{T}}\tilde{\boldsymbol{\epsilon}}_i'\tilde{\boldsymbol{\epsilon}}_j\right)^2-1\right)+\frac{1}{\sqrt{N\left(N-1\right)}}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{1}{TN^{1/2}}\boldsymbol{\ell}_{i,\tau}'\tilde{\mathbf{F}}'\tilde{\mathbf{F}}\boldsymbol{\ell}_{j,\tau}\right)^2+o_p(1)\end{aligned}$$

leading to the desired result.

Proof of Proposition 2

We closely follow the proof of Proposition 1 and obtain similarly

$$\begin{aligned}\widehat{\sigma}_{ij}^2 &= \left(\frac{1}{T}\ddot{\mathbf{u}}_i'\ddot{\mathbf{u}}_j\right)^2+\left(\frac{1}{T}\left(\hat{\boldsymbol{\beta}}_{\tau,i}-\boldsymbol{\beta}_{\tau,i}\right)'\ddot{\mathbf{X}}_i'\ddot{\mathbf{u}}_j\right)^2\\&+\left(\frac{1}{T}\ddot{\mathbf{u}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_{\tau,j}-\boldsymbol{\beta}_{\tau,j}\right)\right)^2+\left(\frac{1}{T}\left(\hat{\boldsymbol{\beta}}_{\tau,i}-\boldsymbol{\beta}_{\tau,i}\right)'\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_{\tau,j}-\boldsymbol{\beta}_{\tau,j}\right)\right)^2\\&-\frac{2}{T^2}\ddot{\mathbf{u}}_i'\ddot{\mathbf{u}}_j\left(\hat{\boldsymbol{\beta}}_{\tau,i}-\boldsymbol{\beta}_{\tau,i}\right)'\ddot{\mathbf{X}}_i'\ddot{\mathbf{u}}_j-\frac{2}{T^2}\ddot{\mathbf{u}}_i'\ddot{\mathbf{u}}_j\ddot{\mathbf{u}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_{\tau,j}-\boldsymbol{\beta}_{\tau,j}\right)+\frac{2}{T^2}\ddot{\mathbf{u}}_i'\ddot{\mathbf{u}}_j\left(\hat{\boldsymbol{\beta}}_{\tau,i}-\boldsymbol{\beta}_{\tau,i}\right)'\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_{\tau,j}-\boldsymbol{\beta}_{\tau,j}\right)\end{aligned}$$

$$+\frac{2}{T^2}\left(\hat{\boldsymbol{\beta}}_{\tau,i}-\boldsymbol{\beta}_{\tau,i}\right)'\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j\ddot{\boldsymbol{u}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_{\tau,j}-\boldsymbol{\beta}_{\tau,j}\right)-\frac{2}{T^2}\left(\hat{\boldsymbol{\beta}}_{\tau,i}-\boldsymbol{\beta}_{\tau,i}\right)'\ddot{\mathbf{X}}_i'\ddot{\boldsymbol{u}}_j\left(\hat{\boldsymbol{\beta}}_{\tau,i}-\boldsymbol{\beta}_{\tau,i}\right)'\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_{\tau,j}-\boldsymbol{\beta}_{\tau,j}\right)$$
$$-\frac{2}{T^2}\ddot{\boldsymbol{u}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_{\tau,j}-\boldsymbol{\beta}_{\tau,j}\right)\left(\hat{\boldsymbol{\beta}}_{\tau,i}-\boldsymbol{\beta}_{\tau,i}\right)'\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_j\left(\hat{\boldsymbol{\beta}}_{\tau,j}-\boldsymbol{\beta}_{\tau,j}\right)$$
$$= \left(\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\right)^2+\sum_{k=1}^{9}\tilde{A}_{k,ij},$$

and also
$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}T\left(\frac{\widehat{\sigma}_{ij}^2}{\widehat{\sigma}_i^2\widehat{\sigma}_j^2}-1\right)=\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{T\left(\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\right)^2}{\sigma_i^2\sigma_j^2}-1\right)+\sum_{k=1}^{10}\tilde{Q}_k$$

where $\tilde{Q}_1,\ldots,\tilde{Q}_{10}$ are defined analogously to the terms in the proof of Proposition 1 but are computed using $\hat{\boldsymbol{\beta}}_{i,\tau}$ rather than a pooled slope coefficient estimator. Thanks to Lemma 1, we obtain that

$$\tilde{\mathcal{T}}=\frac{1}{\sqrt{N(N-1)}}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{T\left(\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\right)^2}{\sigma_i^2\sigma_j^2}-1\right)+o_p(1)$$

under our rate conditions. The result follows using the same arguments as in the proof of Proposition 1.

Proof of Proposition 3

We focus w.l.o.g. on the case of individual-unit estimation. Then, like in the proof of Proposition 2, we have
$$\tilde{\mathcal{T}}_{\tau_k}=\frac{1}{\sqrt{N(N-1)}}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{T\left(\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\right)^2}{\sigma_{ii}^2\sigma_{jj}^2}-1\right)+o_p(1),$$

where the disturbances $u_{i,t,\tau}=\sigma_i\epsilon_{i,t}$ are the same for all τ_k, and therefore

$$\tilde{\mathcal{M}}_K=\frac{1}{N}\sum_{k=1}^{K}\tilde{\mathcal{T}}_{\tau_k}=\frac{1}{\sqrt{N(N-1)}}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{T\left(\frac{1}{T}\ddot{\boldsymbol{u}}_i'\ddot{\boldsymbol{u}}_j\right)^2}{\sigma_{ii}^2\sigma_{jj}^2}-1\right)+o_p(1)$$

for any finite K; the result follows.

Appendix C - Data sources, and additional empirical results and figures

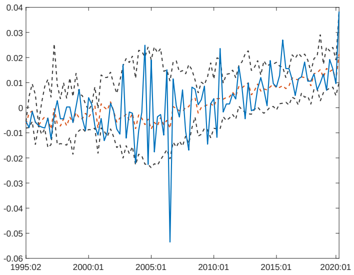
Table C1: Sources of Nominal House Prices Used

Country name	Source	Series	Frequency	sa	Availability
Germany	Deutsche Bundesbank	Residential property prices	annual[a]		1970:Q1 - 2020:Q3
France	Institut National de la Statistique et des Études Économiques (INSEE)	Indice trimestriel des prix des logements anciens - France métropolitaine -	quarterly	yes	1970:Q1 - 2020:Q3
Italy	Eurostat Residential Property Price Index for recent indicator and Nomisma for the past	Eurostat : Residential property prices, existing dwellings, whole country Nomisma : 13 Main Metropolitan Areas - Average current prices of used housing	quarterly semi-annual	no no	1970:Q1 - 2020:Q3
Belgium	Banque National de Belgique	Residential property prices, existing dwellings, whole country	quarterly	no	1970:Q1 - 2020:Q3
Finland	Statistics Finland	Prices of dwellings	quarterly	no	1990:Q3 - 2020:Q3
Ireland	Central Statistics Office	Residential property price index	monthly	no	1970:Q1 - 2020:Q3
Netherlands	Kadaster	House Price Index for existing own homes	monthly	no	1970:Q1 - 2020:Q3
Portugal	European Central Bank	Residential property prices, new and existing dwellings	quarterly	no	1988:Q1 - 2020:Q3
Spain	Banco de España	Precio medio del m2 de la vivienda libre (>2 años de antigüedad)	quarterly	no	1971:Q1 - 2020:Q3
UK	Department for Communities and Local Government	Mix-adjusted house price index	quarterly	no	1970:Q1 - 2020:Q3
US	Federal Housing Finance Agency (FHFA) (from 1991 and OECD adjusted all-transaction index previously)	Purchase and all-transactions indices	quarterly	yes	1970:Q1 - 2020:Q3

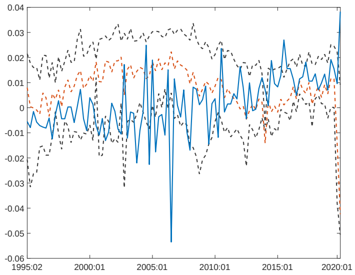
Note: [a] use of quarterly series (owner-occupied apartments in 7 cities) for the quarterly profile.

Table C2: Country specific QR estimation results

		τ								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
DE	$\alpha_{i,\tau}$	0.018***	0.017***	0.025***	0.024***	0.021***	0.019***	0.022***	0.019***	0.014***
	$\beta_{1i,\tau}$	−0.227***	−0.150*	−0.198**	−0.281***	−0.221***	−0.199***	−0.238***	−0.375***	−0.219
	$\beta_{2i,\tau}$	0.025	0.018	−0.002	−0.021	−0.024	−0.003	0.006	0.045*	−0.080***
	$\beta_{3i,\tau}$	0.180	0.146	0.118*	0.046	0.052	0.027	−0.022	0.122*	0.187**
	$\beta_{4i,\tau}$	−0.373***	−0.310***	−0.383***	−0.299***	−0.214***	−0.156***	−0.185***	−0.015	0.228**
	$\beta_{5i,\tau}$	0.031	−0.018	−0.004	−0.048	−0.120**	−0.151***	−0.123**	−0.274***	−0.534***
FR	$\alpha_{i,\tau}$	−0.017	−0.002	−0.012	0.017	0.035**	0.029**	0.029**	0.043***	0.045**
	$\beta_{1i,\tau}$	0.861***	0.928***	0.818***	0.405**	0.270	0.280	0.193	−0.111	−0.170
	$\beta_{2i,\tau}$	0.166***	0.146***	0.082***	0.093***	0.143***	−0.001	−0.026	−0.018	0.018
	$\beta_{3i,\tau}$	0.028	0.165*	0.339***	0.397***	0.302**	0.383**	0.487**	0.545***	0.451
	$\beta_{4i,\tau}$	0.159	−0.027	0.043	−0.226*	−0.368***	−0.292*	−0.272**	−0.405***	−0.339
	$\beta_{5i,\tau}$	−0.351***	−0.205***	0.020	0.072	0.060	0.096	0.109*	0.178***	0.104
IT	$\alpha_{i,\tau}$	0.000	0.005	0.013***	0.012***	0.014***	0.014***	0.017***	0.011***	0.021***
	$\beta_{1i,\tau}$	0.403***	0.216	0.347***	0.337***	0.243***	0.199***	0.097	−0.009	−0.006
	$\beta_{2i,\tau}$	0.080***	0.020	−0.002	0.007	0.029	0.042*	0.048	−0.003	−0.002
	$\beta_{3i,\tau}$	0.288***	0.317***	0.307***	0.350***	0.342***	0.347***	0.381***	0.349***	0.186***
	$\beta_{4i,\tau}$	−0.099	−0.101*	−0.156***	−0.133***	−0.150***	−0.130***	−0.156***	−0.062	−0.085**
	$\beta_{5i,\tau}$	−0.224***	−0.286***	−0.317***	−0.283***	−0.197***	−0.229***	−0.159***	−0.093	−0.170***
ES	$\alpha_{i,\tau}$	−0.005	0.015	0.013	0.011	0.009	0.010*	0.023***	0.026***	0.040***
	$\beta_{1i,\tau}$	0.215	0.505***	0.550***	0.183	0.060	−0.011	−0.189**	−0.186	−0.053
	$\beta_{2i,\tau}$	0.178***	0.172**	0.213***	0.171***	0.165***	0.154***	0.132***	0.145***	0.100***
	$\beta_{3i,\tau}$	0.357**	0.042	0.058	0.198***	0.255***	0.280***	0.199***	0.259***	0.223***
	$\beta_{4i,\tau}$	−0.075	−0.153*	−0.092	−0.040	0.001	0.012	−0.040	−0.060	−0.126***
	$\beta_{5i,\tau}$	−0.306***	−0.097	−0.172	−0.217**	−0.229***	−0.277***	−0.220***	−0.084	0.025
NL	$\alpha_{i,\tau}$	0.003	0.003	0.007	0.008	0.005	0.014***	0.017***	0.023***	0.030***
	$\beta_{1i,\tau}$	0.142	0.139	0.143*	0.124	0.089	−0.001	−0.061	−0.137**	−0.048
	$\beta_{2i,\tau}$	0.153**	0.108***	0.103***	0.082***	0.049*	0.067***	0.080***	0.033	0.010
	$\beta_{3i,\tau}$	0.095	0.213***	0.252***	0.240	0.296***	0.288***	0.346***	0.470***	0.382***
	$\beta_{4i,\tau}$	−0.117	−0.036	−0.034	−0.022	0.076	−0.019	−0.030	−0.080	−0.165
	$\beta_{5i,\tau}$	−0.258	−0.224**	−0.325***	−0.239	−0.176*	−0.183*	−0.203***	−0.231***	−0.146*
IE	$\alpha_{i,\tau}$	0.016*	0.016*	0.007	0.011	0.011*	0.019***	0.030***	0.025***	0.020**
	$\beta_{1i,\tau}$	0.055	0.004	−0.020	0.039	0.075	0.043	0.005	0.013	0.017
	$\beta_{2i,\tau}$	0.114***	0.136***	0.174***	0.174***	0.126***	0.117***	0.071**	0.061**	0.061
	$\beta_{3i,\tau}$	0.137	0.157*	0.120	0.067	0.142	0.142	0.113	0.158*	0.245**
	$\beta_{4i,\tau}$	−0.281***	−0.250**	−0.032	−0.054	0.053	−0.005	−0.003	0.104	0.421***
	$\beta_{5i,\tau}$	−0.595***	−0.427***	−0.223**	−0.127	−0.257**	−0.196	−0.288**	−0.363***	−0.662***
PT	$\alpha_{i,\tau}$	−0.009**	−0.004	0.003	0.005	0.005	0.006	0.011**	0.016**	0.029***
	$\beta_{1i,\tau}$	0.357***	0.341***	0.248**	0.082	0.198	0.181	0.394***	0.230	0.463*
	$\beta_{2i,\tau}$	0.001	0.040	0.046	0.096***	0.137***	0.100***	0.064	0.114*	−0.014
	$\beta_{3i,\tau}$	0.079*	0.032	−0.009	−0.025	−0.021	0.069	−0.003	−0.014	−0.068
	$\beta_{4i,\tau}$	−0.062	−0.066	−0.106**	−0.063	−0.023	0.008	−0.010	−0.013	−0.018
	$\beta_{5i,\tau}$	−0.006	0.014	0.010	−0.019	−0.050	−0.124*	−0.138*	−0.139	−0.289***
BE	$\alpha_{i,\tau}$	0.004	0.000	−0.007	−0.014*	−0.007	−0.005	−0.003	0.007	0.010
	$\beta_{1i,\tau}$	−0.098	−0.077	−0.002	−0.035	0.001	0.110	0.100	0.185	0.039
	$\beta_{2i,\tau}$	−0.036	−0.006	0.049	0.033	0.029	0.018	0.036	0.029	0.053
	$\beta_{3i,\tau}$	−0.017	0.000	0.008	0.025	0.042	0.058**	0.059**	0.086***	0.021
	$\beta_{4i,\tau}$	−0.071	−0.003	0.137	0.251**	0.153	0.143	0.147	0.051	0.129
	$\beta_{5i,\tau}$	−0.145**	−0.046	−0.100*	−0.051	0.101	0.123**	0.086	0.093	−0.020
FI	$\alpha_{i,\tau}$	−0.040***	−0.023***	−0.028***	−0.029***	−0.028***	−0.028***	−0.029***	−0.037***	−0.041***
	$\beta_{1i,\tau}$	0.102	0.003	−0.084**	−0.068	−0.020	0.052	0.063	0.076	0.030
	$\beta_{2i,\tau}$	0.061**	0.056**	0.040*	0.034	0.012	−0.024	−0.041	−0.048*	−0.042
	$\beta_{3i,\tau}$	0.320***	0.250***	0.324***	0.377***	0.185*	0.312***	0.280***	0.381***	0.447***
	$\beta_{4i,\tau}$	0.387***	0.217***	0.294***	0.312***	0.328***	0.351***	0.392***	0.506***	0.623***
	$\beta_{5i,\tau}$	−0.361***	−0.136	−0.112	−0.135	0.057	0.078	0.069	0.002	−0.177
UK	$\alpha_{i,\tau}$	−0.002	0.001	0.004	−0.003	−0.005	0.000	−0.009	−0.012**	0.000
	$\beta_{1i,\tau}$	−0.152	0.122	0.004	−0.216	−0.111	−0.056	−0.029	0.068	0.003
	$\beta_{2i,\tau}$	0.114***	0.149***	0.101***	0.058**	0.058**	0.014	0.002	0.004	−0.003
	$\beta_{3i,\tau}$	0.393	0.290*	0.295***	0.470***	0.491***	0.454***	0.569***	0.600***	0.643***
	$\beta_{4i,\tau}$	−0.166	−0.114	−0.126	0.032	0.102	0.056	0.254***	0.319***	0.156*
	$\beta_{5i,\tau}$	0.038	0.034	0.136**	0.203***	0.230***	0.304***	0.342***	0.352***	0.484***
US	$\alpha_{i,\tau}$	0.017***	0.018***	0.015***	0.014***	0.014***	0.017***	0.016***	0.017***	0.020***
	$\beta_{1i,\tau}$	−0.274***	−0.197***	0.065	0.108**	0.095	0.184***	0.193***	0.166***	0.032
	$\beta_{2i,\tau}$	0.324***	0.269***	0.232***	0.162***	0.159***	0.113***	0.077***	0.046***	0.011
	$\beta_{3i,\tau}$	0.441***	0.297***	0.248***	0.207***	0.282***	0.272***	0.271***	0.312***	0.201***
	$\beta_{4i,\tau}$	−0.376***	−0.243***	−0.195***	−0.126***	−0.084**	−0.081**	−0.038	−0.006	0.041
	$\beta_{5i,\tau}$	−0.129***	−0.154***	−0.122**	−0.107***	−0.148***	−0.184***	−0.187***	−0.207***	−0.197***



(a) DE



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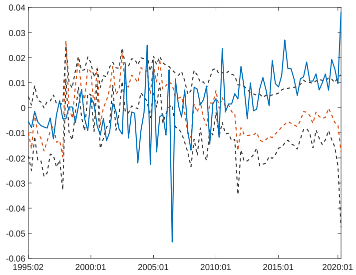

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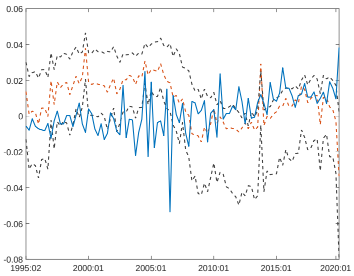

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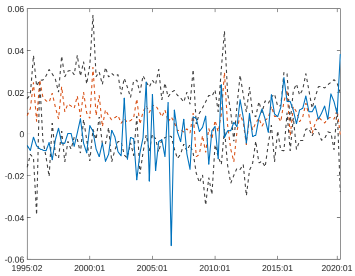
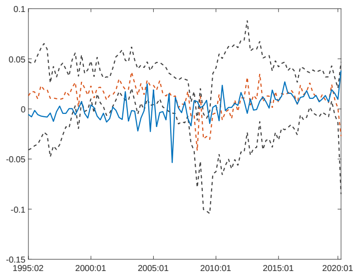
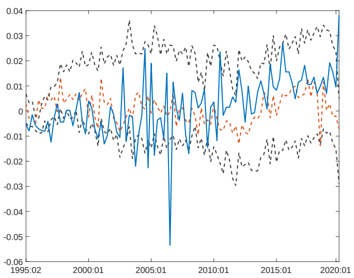


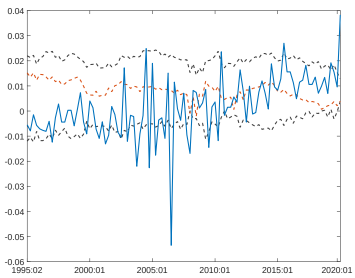
(e) NL



(f) IE



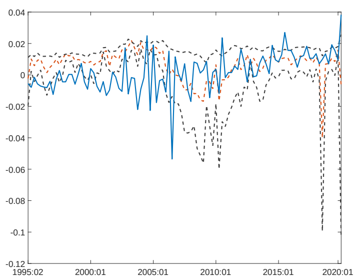
(g) PT



(h) BE

(i) FI

(j) UK



(k) US

90th and 10th percentiles — Median — Δrhp

Figure C1: Quarterly change in log real house prices, conditional median and conditional 10th and 90th percentiles.