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Jörg Franke and Lars P. Metzger[1]

Repeated Contests with Draws

Abstract

We consider a simple contest game with draws where with some probability none of the contestants is selected as winner. If such an outcome occurs, then the contest is repeated in the next period unless either one of the contestants wins the prize or until a final last period is reached. Allowing for finite as well as infinite time horizons and different variations in the timing of effort decisions, the theoretical analysis of this model reveals that the dynamic contest structure has profound implications for intertemporal effort substitution and contest revenue.

JEL-Code: C72, C73, D72

Keywords: Contest theory; repeated contest; dynamic contest; contest with draws

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1 Jörg Franke, University of Bath; Lars P. Metzger, TU Dortmund. *– We are thankful for helpful discussions with Wolfgang Leininger, Marcin Dziubinski, Jaideep Roy, Christian Ewerhart, Cecille Mailler, Diego Moreno, Subhasish Chowdhury, Martin Kaae Jensen and for comments by participants at several conferences and seminars. This project was initiated during a research visit of the first author at the Economic Science Institute at Chapman University and the Haas School of Business at the University of California, Berkeley. Jörg Franke would like to thank Dan Kovenock and the late John Morgan for their hospitality and helpful discussions in the initial phase of this project. The research visit has been partially financed by the Fritz Thyssen Foundation through grant No. 50.15.0.033WW.* *– All correspondence to: Lars P. Metzger, TU Dortmund, Faculty of Business and Economics, Vogelpothsweg 87, 44227 Dortmund, Germany, e-mail: lars.metzger@tu-dortmund.de*

1 Introduction

Contests and prize award schemes are well-established mechanisms to encourage costly contributions and effort from participating agents, comp. McKinsey (2009) and English (2005). Typically, these competitive situations involve agents that compete for an indivisible prize by exerting effort or paying non-refundable bids. Several other applications, like lobbying, selection processes, sport tournaments, crowd-sourcing schemes, as well as conflict and warfare share these characteristics and have been frequently analyzed using methods from contest theory, see Konrad (2009) and Vojnovic (2016).

In some situations, however, the contested prize is actually not awarded (for instance, in the case of a draw, stalemate, tie, or simply due to an idiosyncratic decision by the organizer) and the contest must be repeated in order to determine the winner in a subsequent period. Consider, for instance, selection contests and award schemes, where sometimes none of the candidates/projects is selected, or public procurement, where the organizer reserves and frequently exercises the right of re-tendering. In the US primaries, presidential candidates compete against each other repeatedly in different caucuses until one candidate obtains a majority of delegates, while lobbyists have to decide which decision makers to target in multi-stage legislative processes involving several political bodies. Moreover, a number of sports tournaments like tennis, volleyball, or soccer involve tie-breaks or penalty shootouts, where the contest (game or penalty) is repeated as long as the final winner is determined. Naturally, strategic agents that participate in these types of contest games should be aware of the possibility that the contest might fail to be decided in a specific round and has to be repeated. The resulting dynamic structure of these contests affects agents' inter-temporal effort allocation and finally also the resulting contest revenue. Hence, analyzing the underlying dynamics is not only important for individual agents but also for revenue-interested contest organizers.

In this paper we consider a simple contest game framework with the described dynamic structure; that is, we allow for a non-decisive contest outcome in the sense that none of the contestants might win the prize in a given period with a specific probability. If such a non-decisive outcome actually occurs, then the contest is repeated in the next period unless either one of the contestants wins the prize, or until a final last period is reached. This dynamic structure of potentially repeated contests with draws leads to intertemporal effort substitution by the agents, which is highly dependent on the underlying fundamentals of the contest. We specifically demonstrate that the time horizon (finite or infinite) as well as the timing of the effort decision (ex-ante decision or period-specific effort) have profound implications for individual effort exertion and contest revenue.

Our model comprises several variants of the underlying basic setup to trace out these dependencies. With respect to the timing of effort decisions we consider two different alternatives: In Model A, contestants decide ex-ante how much to invest in each period, while in Model B the effort decision is made in the specific period and can therefore be conditioned on the fact whether the respective period has been reached. With respect to the time horizon, we allow for both a finite as well as an infinite time horizon where a finite time horizon means that there exists a finite period T in which the probability of a draw is zero. In other words, at the latest in period T one of the contestants wins the contest for sure. Under an infinite time horizon, in contrast, the contest game can potentially go on forever. In order to be able to isolate the dynamic effects of intertemporal effort substitution, we also specify a benchmark model where we restrict agents' decision to be time-independent, that is, constant between periods. Hence, in the benchmark model intertemporal effort substitution is switched off by design.

Given the specific structure of the contest game, the theoretical analysis is based on the concept of (subgame-perfect, when appropriate) Nash-equilibrium with (potentially) multi-dimensional strategy space. Starting with the benchmark model, where intertemporal substitution of effort is ruled out by assumption, we find equivalence with the standard one-stage lottery contest game without draws. Hence, any deviation from the benchmark in one of the four dynamic variants of our setup can be attributed to strategic intertemporal effort substitution. However, our analysis of the first variant, Model A with finite time horizon, actually shows that intertemporal effort substitution does not change contest revenue: Although contestants substitute effort intertemporally in the sense that effort is exerted only in the first and the last period (which is increasing, resp. decreasing in the probability of a draw), aggregate effort in total is the same as in the benchmark model. Hence, contestants basically substitute some effort from the first into the last period, which leaves overall effort exertion and contest revenue unaltered.

Equilibrium effort allocation in Model A with an infinite time horizon is remarkably different. Without the existence of a last period, contestants exert positive effort in each period which is decreasing in time and converges to zero. Hence, the probability of a draw in a specific period is increasing over time. We show that in equilibrium a 'perpetual' draw occurs with positive probability and characterize the equilibrium in explicit form. We then demonstrate that contest revenue can be higher or lower than in the benchmark model and identify the respective conditions.

In Model B with a finite time horizon effort decisions are made in each respective period. Hence, we rely on backward induction to characterize the subgame-perfect equilibrium. Based on a difference equation that characterizes equilibrium effort for two arbitrary but subsequent time periods, we are able to demonstrate that per period effort is bounded between two specific finite values and is decreasing except in the last period. This difference equation is also instrumental in showing that contest revenue is higher than in the benchmark model. For Model B with an infinite time horizon the same difference equation holds in equilibrium. However, in contrast to the situation with a finite time horizon the equilibrium effort must be stationary with an infinite time horizon which implies that effort exertion is at the previously determined upper bound. Calculating the expected contest revenue for this variant leads to the conclusion that rent dissipation is higher than in the benchmark model. Our results from the theoretical analysis also allow us to establish a revenue-ranking along the 2×2-dimensions considered in our framework and to identify the variant that is revenue-dominant irrespectively of the underlying parameter values.

Related Literature

Our framework allows for a non-decisive contest outcome in each period in the sense that potentially none of the players wins the contest in a given period. To capture this possibility, we rely on a modified lottery contest in the style of Tullock (1980), augmented by the possibility of a draw as proposed and axiomatized by Blavatskyy (2010). In this specification the win probabilities of all players do not sum up to one such that the remaining probability mass can be attributed to the event of a non-decisive outcome. This also implies that the draw probability becomes endogenous, more specifically, decreasing in total effort levels. Several studies by now apply this class of contest success functions to analyze the strategic implications of the possibility of draws in static one-period contest games, see for instance, Li et al. (2023), Minchuk (2022), Gama and Rietzke (2019), Deng et al. (2018), Dasgupta and Nti (1998), and Nti (1997). The analysis of a dynamic setup, where the realization of a draw leads to a repetition of the contest in the next period which is the focus of our analysis, has to our knowledge not been addressed in the literature.

The specific way in which the draw probability is typically modeled in these types of contest games allows for different potential interpretations and applications. We here briefly elaborate on three interpretations to motivate the specific modeling assumption regarding the draw probability. Firstly, the draw parameter can be interpreted as a stochastic minimum bid or reserve price, which is not exactly known by the contestants.[1] The higher the exerted

[1] Chowdhury (2017) considers a similar framework with a stochastic minimum bid re-

effort by the contestants, the higher the probability that the reserve is met (in other words, the probability of a draw is decreasing in contestants' effort). In case that the reserve is not met, the contest is called off and has to be repeated in the next period. Secondly, from the perspective of the contestants it is as if another hypothetical but non-strategic contestant is participating in the contest.[2] If this hypothetical player wins, then a draw is realized and none of the other players wins the contest. Thirdly, there exist a number of micro-foundations for the static contest game with draws resulting in the specific functional form that we apply in our setup. In the innovation game of Loury (1979), for instance, a draw occurs if none of the firms is able to make an innovation during the process. In the underlying continuous time discovery process the draw probability depends on the discount factor (or alternatively, the speed of the innovation process). In the rank-order tournament of Jia (2012), the performance (but not the effort) of a contestant is observable by the organizer. Moreover, the organizer requires a specific margin between the observable outputs of two competing players in order to rank them.[3] The degree of this margin then affects the draw probability in the reduced one-stage contest with draws.

As our setup embeds a simple contest with draws into a dynamic framework, it is also related to the literature on dynamic contests. In most contributions from this strand of the literature, effort is (either simultaneously or sequentially) exerted in each period and 'accumulates' over time as in races or tug-of-wars, see Konrad (2009, chap. 8) and Vojnovic (2016, chap. 7). In those models equilibrium effort is typically history-dependent due to the specific structure of the model setup. Model B of our framework shares this feature of history-dependent equilibrium behavior although the ex-ante probability to win the contest in a given period does not depend on previous effort exertion. Hence, the dynamics in our model are due to the endogenous continuation/draw probability[4] and not due to the assumption that effort

quirement in a two-player all-pay auction with complete information. If both players do not meet the minimum bid, then none of the players obtains the prize.

[2]In a selection contest the effort of the hypothetical non-strategic player could be interpreted as the benchmark effort level that a contest organizer typically expects from a fictional benchmark player. If effort levels of candidates fall short with respect to the expectations of the contest organizer, then the organizer is less likely to select any of the candidates and instead might prefer to repeat the contest in the next round.

[3]For contributions that explicitly model draws as ties in an all-pay auction setup with complete information in the sense that no player wins if the bids are either equal or too close to each other, see Gelder et al. (2022), or Szech (2015).

[4]In this sense our framework is not a repeated game in the strict game-theoretical sense because the stage game is repeated with an (endogenously determined) probability. This property implies that standard folk theorems cannot be applied in our framework.

accumulates over time.

The rest of the paper is structured as follows. In section 2 we describe the formal model and the considered variants regarding finite or infinite time horizon and the timing of the effort decision. In section 3, we analyze Model A where effort decisions are made before the actual contest game starts, while section 4 deals with Model B where effort decisions for a given period are made in the specific period. Section 5 provides a revenue-ranking along the two dimensions covered in our framework and section 6 concludes.

2 Model

We consider a dynamic framework of repeated contests with draws, where $n \geq 2$ players compete for a single prize of common value $v > 0$. Each player $i = 1, \dots, n$ can affect its probability p_i^t to win the prize in a given period $t = 1, 2, \dots, T$ with $T \geq 2$ by exerting period-specific effort $x_i^t \geq 0$. Irrespectively of winning or not, player i faces the costs x_i^t of exerting effort. In contrast to standard contest models, a contest in period $t < T$ can result in a non-decisive outcome (draw) which happens with probability p_r^t. In this case the contest is repeated in period $t+1$ and the effort cost from period t are sunk. If a contest continues until the last period T, then the probability of a draw is, by assumption, zero; however, we also consider variants of our model where no finite period exists (i.e., $T = \infty$); that is, the contest could potentially continue forever.

The probabilities p_i^t and p_r^t depend on the individual investments $x_i^t, \dots, x_n^t$ and on the parameter $r \in (0, v)$ in the following way:

$$p_i^t = \begin{cases} \frac{x_i^t}{\sum_{j=1}^n x_j^t + r} & \text{if } t < T \\ \frac{x_i^t}{\sum_{j=1}^n x_j^t} & \text{if } t = T \text{ and } \sum_{j=1}^n x_j^t > 0 \\ \frac{1}{n} & \text{if } t = T \text{ and } \sum_{j=1}^n x_j^t = 0 \end{cases}$$

$$p_r^t = \begin{cases} \frac{r}{\sum_{j=1}^n x_j^t + r} & \text{if } t < T \\ 0 & \text{if } t = T \end{cases}$$

This specification of a simple modified Tullock lottery contest with draws is in line with Blavatskyy (2010) and Jia (2012)[5] and has sufficient tractability for a dynamic analysis. For a justification and motivation of this specific

[5]Our specification is a special case of the class of CSFs characterized in Theorem 2 of Jia (2012) where $c = r/(n-1)$ and $g(x_i) = x_i$ for all i, as well as Corollary 2 of Blavatskyy (2010) where $\alpha_i = r = 1$ for all i.

functional form assumption we refer to the corresponding paragraphs in section 1.

We consider two alternative variants of the basic setup that differ with respect to the point in time when the effort decision is made, or respectively, the effort costs are incurred.[6] In Model A all players simultaneously make their decision (and incur all their respective effort costs) in period 1 regardless of whether any subsequent periods have been reached or not. Hence, the resulting contest game is a one-stage simultaneous-move game with multi-dimensional individual strategy space. In Model B all players make their effort decision (and incur their effort costs for the respective period) in a specific period if it has been actually reached, that is, if there has been a draw in all previous periods. In this case the resulting game is a multi-stage (extensive form) game with perfect information. For both model variants we also allow for a finite and infinite time horizon: In the former case there exists a last period (without draw), while in the latter case a draw in each period cannot be ruled out ex-ante. The resulting 2×2-structure of our framework is represented in Table 1.

	Finite Time Horizon $T < \infty$	Infinite Time Horizon $T = \infty$
Model A $(x_i^1, x_i^2, \ldots, x_i^T)$ in $t = 1$	Section 3.1	Section 3.2
Model B x_i^t in $t = 1, \ldots T$	Section 4.1	Section 4.2

Table 1: Lottery Contest Framework

3 Analysis of Model A

As we are interested in the dynamics of effort exertion over time, we first construct a benchmark model where inter-temporal effort substitution is shut off

[6] An alternative interpretation of these differences in timing relates to the commitment power of the players: In Model A players have commitment power with respect to their effort decisions for all future periods, while in Model B they are able to renegotiate their effort decision (which implies they will refrain form exerting effort in subsequent periods $s = t + 1, \ldots, T$ if the contest has been terminated in a specific period t).

by assumption; in other words, a player's individual effort does not vary between periods. Analyzing this benchmark model and comparing the respective result with the full-fledged model allows us to analyze how equilibrium behavior is affected by the dynamic structure in the different variants of Model A.

Benchmark Model

In the benchmark model players decide simultaneously and incur the costs for time-invariant effort x_i of each period $t = 1, \ldots, T$ in the first period. Hence, for each period $t < T$ the probability for winning and for a draw is time-invariant as well: $p_i = \frac{x_i}{\sum_{j=1}^n x_j + r}$, $p_r = \frac{r}{\sum_{j=1}^n x_j + r}$ while $p_i^T = \frac{x_i}{\sum_{j=1}^n x_j}$. Each player i maximizes its expected payoff $E[u_i(\mathbf{x})]$ with $\mathbf{x} = (x_1, \ldots, x_n)$, which can be calculated as the summed up probability to win the prize in a specific period conditional on the respective period being reached subtracted by the costs of effort exertion for all periods.

With a finite time horizon $T < \infty$ the expected payoff function can then be expressed as follows, where the total effort of player $i = 1, \ldots, n$ is denoted by $X_i = \sum_{t=1}^T x_i = T \cdot x_i$:

$$\begin{aligned} E[u_i(\mathbf{x})] &= v \cdot p_i - x_i + p_r \cdot v \cdot p_i - x_i + (p_r)^2 \cdot v \cdot p_i - x_i + \ldots \\ &\quad \ldots + (p_r)^{T-2} \cdot v \cdot p_i - x_i + (p_r)^{T-1} \cdot v \cdot p_i^T - x_i \\ &= v \cdot p_i \sum_{t=0}^{T-2} (p_r)^t + v \cdot p_i^T \cdot (p_r)^{T-1} - X_i \end{aligned}$$

With an infinite time horizon $T = \infty$ there is no last period. In this case the finite sum in the expected payoff function is substituted by an infinite sum:

$$\begin{aligned} E[u_i(\mathbf{x})] &= v \cdot p_i - x_i + p_r \cdot v \cdot p_i - x_i + (p_r)^2 \cdot v \cdot p_i - x_i + \ldots \\ &= v \cdot p_i \sum_{t=0}^{\infty} (p_r)^t - X_i \end{aligned}$$

The following result demonstrates that there exists a unique equilibrium in the benchmark model with a finite and an infinite time horizon. The proof is based on the observation that the benchmark model is equivalent to a slightly modified standard one-stage lottery contest without draws.

Proposition 1 *In the benchmark model total individual effort and contest revenue in the unique equilibrium are as follows:*

$$X_i = \frac{n-1}{n^2} v, \qquad X = n \cdot X_i = \frac{n-1}{n} v.$$

PROOF : With a finite time horizon the expected payoff function can be simplified by observing that the formula contains a geometric series and by using the relations $\frac{p_i}{1-p_r} = p_i^T = \frac{x_i}{\sum_{j=1}^n x_j} = \frac{X_i}{\sum_{j=1}^n X_j}$:

$$\begin{aligned} E[u_i(\mathbf{x})] &= v \cdot p_i \sum_{t=0}^{T-2} (p_r)^t + v \cdot p_i^T \cdot (p_r)^{T-1} - X_i \\ &= v \cdot p_i \frac{1-(p_r)^{T-1}}{1-p_r} + v \cdot p_i^T \cdot (p_r)^{T-1} - X_i \\ &= v \cdot p_i^T \left(1-(p_r)^{T-1}\right) + v \cdot p_i^T \cdot (p_r)^{T-1} - X_i \\ &= v \cdot p_i^T - X_i \\ &= v \frac{X_i}{\sum_{j=1}^n X_j} - X_i \end{aligned}$$

With an infinite time horizon the expected payoff function simplifies as follows:

$$\begin{aligned} E[u_i(\mathbf{x})] &= v \cdot p_i \sum_{t=0}^{\infty} (p_r)^t - X_i \\ &= v \cdot p_i \frac{1}{1-p_r} - X_i \\ &= v \frac{X_i}{\sum_{j=1}^n X_j} - X_i \end{aligned}$$

Note, that the last equations in both cases are identical and coincide with a standard one-stage lottery contest framework (without draws). Hence, well-established results from contest theory lead to the expressions in the proposition. □

From the perspective of a revenue-interested contest organizer this result has a straight-forward implication. If players are not able to adopt to the dynamic structure of the contest (i.e., they are restricted to exert time-invariant effort), then they will spread the same amount of effort that they typically exert in a standard one-stage lottery contest without draws evenly over all periods. Hence, in the benchmark model a contest organizer cannot benefit from implementing a dynamic structure of repeated conflicts with draws because total effort and revenue are not affected by the induced dynamics. The following sections will reveal whether this implication is also robust under the more realistic assumption of time-variant effort exertion by the players.

If players are able to use time-dependent effort in the framework of Model A, then each player $i = 1, \ldots, n$ chooses an effort vector $\mathbf{x_i} = (x_i^1, x_i^2, \ldots, x_i^T)$ with $x_i^t \geq 0$ for all periods in period 1 and also faces the respective costs of effort exertion for all periods in period 1. Expected payoff can then be expressed in the following way (using the convention that $\prod_{s=1}^{0} p_r^s = 1$):

$$E[u_i(\mathbf{x_i}, \mathbf{x_{-i}})] = \sum_{t=1}^{T} \left(v \cdot p_i^t \cdot \prod_{s=1}^{t-1} p_r^s - x_i^t \right).$$

This expected payoff function is strictly concave in player i's effort (see Proposition 9 in the appendix). Assuming that the equilibrium is characterized by first-order conditions, the system of first-order conditions can be reformulated as follows (using the convention $\sum_{k=t+1}^{t} p_i^k = 0$, as well as the fact that $\frac{\partial p_i^t}{\partial x_i^t} = (1-p_i^t)\frac{p_i^t}{x_i^t}$ for $t \leq T$ and $\frac{\partial p_r^t}{\partial x_i^t} = -\frac{\left(p_r^t\right)^2}{r}$ for $t < T$):

$$\begin{aligned} \frac{\partial E[u_i(\mathbf{x_i}, \mathbf{x_{-i}})]}{\partial x_i^t} &= \frac{\partial p_i^t}{\partial x_i^t} \cdot v \prod_{s=1}^{t-1} p_r^s + \sum_{k=t+1}^{T} p_i^k \cdot v \prod_{s=1}^{k-1} p_r^s \frac{1}{p_r^t} \frac{\partial p_r^t}{\partial x_i^t} - 1 \leq 0 \\ \Leftrightarrow \quad & \frac{p_i^t}{x_i^t} \left(\prod_{s=1}^{t-1} p_r^s - \sum_{k=t}^{T} p_i^k \prod_{s=1}^{k-1} p_r^s \right) \leq \frac{1}{v}, \end{aligned} \tag{1}$$

where the first-order condition is satisfied by equality if $x_i^t > 0$ in period $t = 1, \ldots, T$.[7] It should be noted at this point, that any $r \geq v$ implies that in equilibrium players will only exert effort in the last period or exert no effort at all (see Lemma 14 in the appendix). Hence, we restrict the subsequent analysis to the more interesting cases where $r \in (0, v)$.

The following result establishes equilibrium symmetry in Model A with finite and infinite time horizon. The proof is standard and therefore relegated to the appendix.

Lemma 1 *In model A any equilibrium must be symmetric.*

Given equilibrium symmetry and for notational convenience we suppress the player index i from now on for any vector satisfying inequality (1); that is, we define $x^t = x_1^t = \ldots = x_n^t$ and $p^t = p_1^t = \ldots p_n^t$ for all $t = 1, \ldots, T$.

[7] Note, that eq. (1) is well-behaved with the exception of $x_1^T = \ldots = x_n^T = 0$ (because $\frac{p_i^t}{x_i^t} = \frac{p_j^t}{x_j^t} = \frac{p_r^t}{r} = \frac{1}{\sum_{k=1}^{n} x_k + r}$ for $t < T$ and $\frac{p_i^t}{x_i^t} = \frac{p_j^t}{x_j^t} = \frac{1}{\sum_{k=1}^{n} x_k}$ for $t = T$). This is a typical issue in Tullock contests due to the discontinuity of the CSF at this singular point. However, this specific vector of effort levels is never part of an equilibrium strategy as our subsequent analysis reveals.

3.1 Finite Time Horizon

In Model A with a finite time horizon one of the players wins the contest for sure, at the latest in the last period. Hence, from the perspective of a player two periods are of specific interest: The last period is special in the sense that (given it is actually reached) effort is not wasted because the probability of a draw is zero in this period. Moreover, the first period is special in the sense that it is reached for sure (in contrast to all other periods that require the occurrence of draws in all previous periods). The following result demonstrates that this comparative advantage of exerting effort in the first and last period implies that players are generally only willing to exert positive effort in these two periods, refraining from any effort exertion in the intermediate periods $t = 2, \dots, T-1$.

Proposition 2 *In Model A with a finite time horizon players only exert positive effort in the first (provided the draw parameter r is sufficiently low) and the last period. Equilibrium effort can be characterized as follows:*

$$\begin{aligned} x^1 &= \begin{cases} \frac{n-1}{n^2}v - \frac{r}{n} & \text{if } r < \frac{n-1}{n}v \\ 0 & \text{if } r \geq \frac{n-1}{n}v, \end{cases} \\ x^t &= 0 \ \text{ for all } t = 2, \dots, T-1, \\ x^T &= \begin{cases} \frac{r}{n} & \text{if } r < \frac{n-1}{n}v \\ \frac{n-1}{n^2}v & \text{if } r \geq \frac{n-1}{n}v. \end{cases} \end{aligned}$$

Proof : According to Lemma 1 an equilibrium is symmetric. Moreover, it holds by definition for any period t that either one of the players already won the contest previously or there must have been a draw in all previous periods. This statement can be formally expressed as follows:

$$n \sum_{k=1}^{t-1} p_k \prod_{s=1}^{k-1} p_r^s + \prod_{s=1}^{t-1} p_r^s = 1 \Leftrightarrow \sum_{k=1}^{t-1} p_k \prod_{s=1}^{k-1} p_r^s = \frac{1}{n} - \frac{1}{n} \prod_{s=1}^{t-1} p_r^s \ .$$

In period T there is no draw and hence

$$n \sum_{k=1}^{T} p_k \prod_{s=1}^{k-1} p_r^s = 1 \Leftrightarrow \sum_{k=1}^{T} p_k \prod_{s=1}^{k-1} p_r^s = \frac{1}{n} \ . \tag{2}$$

Subtracting both equations from each other leads to the following equation:[8]

$$n\sum_{k=t}^{T} p_k \prod_{s=1}^{k-1} p_r^s = \prod_{s=1}^{t-1} p_r^s \Leftrightarrow \sum_{k=t}^{T} p_k \prod_{s=1}^{k-1} p_r^s = \frac{1}{n}\prod_{s=1}^{t-1} p_r^s$$

which we substitute in equation (1) to simplify the first order condition with $\frac{p_r^t}{r} = \frac{p_i^t}{x_i^t}$ for $i = 1, \ldots, n$ and any period $t < T$:

$$\prod_{s=1}^{t} p_r^s \leq \frac{r}{v}\frac{n}{n-1} \tag{3}$$

Evaluating this modified first-order condition for period $t = 1$ implies that $x^1 = \frac{n-1}{n^2}v - \frac{r}{n}$ if $r < \frac{n-1}{n}v$ and $x^1 = 0$ if $r \geq \frac{n-1}{n}v$. Based on these values the probability for a draw in period 1 can be calculated as $p_r^1 = \frac{n}{n-1}\frac{r}{v}$ if $r < \frac{n-1}{n}v$ and $p_r^1 = 1$ if $r \geq \frac{n-1}{n}v$. Evaluating the modified first-order condition for period $t = 2$ and using the previous results implies that $x^2 = 0$ and, accordingly, $p_r^2 = 1$, which also holds for the subsequent periods $t = 3, \ldots, T-1$. For period T we get via (1) and by using our previous results

$$\frac{1}{n \cdot x^T}\prod_{s=1}^{T-1} p_r^s \frac{n-1}{n} \leq \frac{1}{v} \Rightarrow x^T = \begin{cases} \frac{r}{n} & \text{if } r < \frac{n-1}{n}v \\ \frac{n-1}{n^2}v & \text{if } r \geq \frac{n-1}{n}v \end{cases}.$$

□

Proposition 2 implies that the extent of effort exertion in the first and in the last period depends on parameter r which governs the likelihood of a draw. The higher the probability of a draw, the more reluctant are players to exert effort in the first period (because it is more likely that their effort is wasted) and the more they are willing to exert effort in the last period (which is reached with a high probability and where effort is more decisive because there is no draw). Hence, there is clearly intertemporal effort substitution at work between the first and the last period. The following result however shows that overall effort exertion is not affected: Players' total effort is as high as in the benchmark model but now allocated selectively to the first and last period.

[8] This equation has a straight-forward interpretation: The conditional probability that one of the players is going to win the contest in periods $t, t+1, \ldots, T-1, T$, given that period t has been reached, is equal to one. Hence the unconditional probability that one of the players is going to win the contest in periods $t, t+1, \ldots, T-1, T$ is equal to the probability that period t has been reached which happens if there was a draw in all previous periods.

Corollary 1 *Total effort exertion in Model A with finite time horizon is the same as total effort exertion in the Benchmark Model:*

$$\sum_{i=1}^{n}\sum_{t=1}^{T} x_i^t = \frac{n-1}{n} v.$$

3.2 Infinite Time Horizon

In Model A with an infinite time horizon players face an infinite stream of sunk period costs for exerting effort in all periods (irrespectively of whether the respective period has been actually reached or not) while the prize has a finite value. This suggests that in equilibrium players will either exert zero effort in most periods (as in the previous section with a finite time horizon), or will choose positive effort levels that converge very fast to zero. The following results show that the second intuition holds in equilibrium implying that with positive probability the contest continues forever (with a 'perpetual draw') because the probability of a draw converges to one in later periods. Hence, equilibrium behavior is qualitatively very different in these two variants of Model A that only differ with respect to the finiteness of the time horizon.

Proposition 3 *In Model A with infinite time horizon players exert (i) positive effort in each period, which is (ii) decreasing over time and converges to zero in the long run, resulting in (iii) a perpetual draw with positive probability. (iv) Equilibrium effort can be characterized as follows:*

$$x_i^t = \frac{n-1}{n^2}(v-r)\frac{r}{v(n^{t-1}-1)+r} \quad \textit{for } t = 1, 2, \ldots.$$

The proof of Proposition 3 proceeds in a series of lemmata, where each statement (i) – (iv) is proved in a separate lemma.[9]

Lemma 2 *In equilibrium there is positive effort exertion in each period.*

Proof : The proof proceeds in two steps. In the first step we derive a property that must be satisfied in any equilibrium with zero effort exertion in (at least) one period. In the second step we prove that this property is not

[9]In contrast to the finite time horizon addressed in the previous subsection, the proof is more complex due to the possibility of a perpetual draw where none of the players wins the overall contest game (in other words, we cannot use the convenient reformulation from the previous subsection, where equation (2) was applied to obtain the simplified explicit equilibrium characterization in (3)).

consistent with the equilibrium notion. Hence, an equilibrium cannot entail zero effort exertion in any period.

Step 1. If $x_i^t = 0$ in equilibrium, then $x_i^{t+1} = 0$.
Proof: By Lemma 1 $x_i^t = x^t$ for all $i = 1, \ldots, n$ and $t = 1, 2, \ldots$. The rest of the proof proceeds by contradiction. Suppose $x^t = 0$ and $x^{t+1} > 0$ in equilibrium for some t. Then since $x^{t+1} > 0$ and $p_r^t = 1$ the binding inequality (1) for period $t+1$ implies

$$\frac{p_r^{t+1}}{r}\left(\prod_{s=1}^{t-1} p_r^s - \sum_{k=t+1}^{\infty} p^k \prod_{s=1}^{k-1} p_r^s\right) = \frac{1}{v},$$

where we used the fact that $\frac{p^{t+1}}{x^{t+1}} = \frac{p_r^{t+1}}{r} = \frac{1}{n \cdot x^{t+1} + r}$.

Since $x^t = 0$ and $p^t = 0$ by assumption, inequality (1) for period t implies

$$\frac{1}{r}\left(\prod_{s=1}^{t-1} p_r^s - \sum_{k=t+1}^{\infty} p^k \prod_{s=1}^{k-1} p_r^s\right) \leq \frac{1}{v} \ .$$

Combining these two statements implies $p_r^{t+1} \geq 1$ and hence $x^{t+1} \leq 0$, a contradiction.

The contraposition of the statement from Step 1 is $x^{t+1} > 0 \Rightarrow x^t > 0$. Hence, in any equilibrium there is positive effort exertion up to some period $\tau \geq 0$, where τ does not need to be finite, and zero effort exertion thereafter:

$$x^* = \left(x^{1*}, x^{2*}, \ldots, x^{\tau*}, 0, 0, \ldots\right) \ . \tag{4}$$

In the next step we show that a finite τ is, however, not consistent with the notion of x^* being an equilibrium.

Step 2. In equilibrium there cannot be zero effort exertion in any period.
We show by contradiction that an equilibrium cannot entail zero effort exertion in some period $t = 1, \ldots$ by ruling out equilibria of the class characterized in equation (4) with a finite τ. Suppose there exists some finite $\tau \geq 1$ such that $x^t > 0$ for all $t \leq \tau$ and $x^t = 0$ for all $t > \tau$. With $\frac{p^\tau}{x^\tau} = \frac{p_r^\tau}{r}$ and $\sum_{k=\tau}^{\infty} p^k \prod_{s=1}^{k-1} p_r^s = p^\tau$ equation (1) implies for period τ that

$$\frac{p_r^\tau}{r}\left(\prod_{s=1}^{\tau-1} p_r^s - p^\tau\right) = \frac{1}{v} \Leftrightarrow \prod_{s=1}^{\tau} p_r^s = \frac{r}{v} + p_r^\tau \cdot p^\tau,$$

and for period $\tau+1$ with $\frac{p^{\tau+1}}{x^{\tau+1}} = \frac{p_r^{\tau+1}}{r} = \frac{1}{r}$ and $\sum_{k=\tau+1}^{\infty} p^k \prod_{s=1}^{k-1} p_r^s = 0$ that

$$\frac{1}{r}\left(\prod_{s=1}^{\tau} p_r^s - 0\right) \leq \frac{1}{v},$$

which in combination implies $p_r^\tau \cdot p^\tau \leq 0$, a contradiction. □

From Lemma 2 we know that equilibrium effort is positive in each period which implies that equation (1) is binding for any two consecutive periods t and $t+1$. We reformulate this equation for period t as follows:

$$\sum_{k=t}^{\infty} p^k \prod_{s=1}^{k-1} p_r^s = \prod_{s=1}^{t-1} p_r^s - \frac{1}{v}\frac{x^t}{p^t},$$

and substitute this expression in equation (1) evaluated for period $t+1$

$$v \prod_{s=1}^{t-1} p_r^s \left(p_r^t - 1 + p^t\right) = \frac{x^{t+1}}{p^{t+1}} - \frac{x^t}{p^t} \; .$$

As the equilibrium is symmetric, we use $p_r^t + n \cdot p^t = 1$ and the fact that $\frac{x^{t+1}}{p^{t+1}} - \frac{x^t}{p^t} = n \cdot x^{t+1} - n \cdot x^t$ to simplify further:

$$(n-1)v \prod_{s=1}^{t-1} p_r^s \cdot p^t = n \cdot x^t - n \cdot x^{t+1} \Leftrightarrow$$

$$x^{t+1} = x^t \left(1 - \frac{n-1}{n}\frac{v}{r}\prod_{s=1}^{t} p_r^s\right) \; . \tag{5}$$

The difference equation in (5) completely characterizes the equilibrium for all periods. Note, that the expression on the right-hand-side does not only depend on x^t but also (through $\prod_{s=1}^{t} p_r^s$) on all previous effort levels x^s for all $s \leq t$. This difference equation is crucial to determine the results in the following lemmata.

Lemma 3 *Equilibrium effort declines over time and converges to zero.*

Proof : The expression in brackets on the right-hand-side of equation (5) is clearly less than one. By Lemma 2, $x_t > 0$ for all $t < \infty$, therefore the expression in brackets is positive. This implies that effort is declining over periods, hence there is no stationary $\mathbf{x}$ with $x^t > 0$. Moreover, an inspection of the difference equation reveals that $\mathbf{x} = 0$ is the unique stationary point which coincides with the infimum. As equilibrium effort is declining over time the series must converge to this infimum. □

The preceding lemma implies that the probability of a draw in each period is positive, increasing over time, and converges to one for later periods. This suggests that the overall contest game could actually lead to a 'perpetual draw', where none of the players wins the contest game although all players exert positive effort in each period.[10] The following result demonstrates that this type of perpetual draw actually occurs in equilibrium with positive probability.

Lemma 4 *In equilibrium, the probability of an infinite draw is given by*

$$P := \prod_{s=1}^{\infty} p_r^s = \frac{r}{v} \in (0,1) \ .$$

PROOF : By Lemma 2 equation (1) is binding for period $t = 1$. With $n \sum_{t=1}^{\infty} p^t \prod_{s=1}^{t-1} p_r^s + P = 1$ we get

$$x^1 = \frac{1}{n} \left(\frac{n-1}{n} v - r + \frac{v}{n} P \right)$$

and

$$p_r^1 = \frac{r}{v} \cdot \frac{n}{n-1+P} \ .$$

With equation (5) we get

$$x^2 = \frac{1}{n} P \left[\frac{v}{n} - \frac{r}{n-1+P} \right] = \frac{v}{n^2} P \left[1 - p_r^1 \right]$$

and

$$p_r^2 = \frac{r}{\frac{v}{n} P \left[1 - p_r^1 \right] + r} = \frac{r}{v} \cdot \frac{n}{P + (n-1) p_r^1} \ .$$

Suppose that

$$p_r^t = \frac{r}{v} \cdot \frac{n}{P + (n-1) \prod_{s=1}^{t-1} p_r^s}$$

holds for some t. Then we have

$$\frac{1}{p_r^t} = \frac{v}{r} \cdot \frac{P + (n-1) \prod_{s=1}^{t-1} p_r^s}{n}$$

[10]This type of equilibrium behavior, where players exert costly effort in each period although the equilibrium outcome is an ongoing stalemate, has also been observed in other contexts, for instance, in the 'dollar auction game', where players can sequentially increase their previous (costly) bids until one of them gives up, see Leininger (1989).

and

$$x^t = \frac{r}{n}\left(\frac{1}{p_r^t} - 1\right) = \frac{v}{n^2}\left((n-1)\prod_{s=1}^{t-1} p_r^s + P\right) - \frac{r}{n}$$

and with equation (5) we have

$$x^{t+1} = \left(\frac{v}{n^2}\left((n-1)\prod_{s=1}^{t-1} p_r^s + P\right) - \frac{r}{n}\right)\left(1 - \frac{n-1}{n}\frac{v}{r}\prod_{s=1}^{t} p_r^s\right)$$

and

$$\begin{aligned}
p_r^{t+1} &= \frac{r}{nx^{t+1} + r} \\
&= \frac{r}{\left(\frac{v}{n}\left((n-1)\prod_{s=1}^{t-1} p_r^s + P\right) - r\right)\left(1 - \frac{n-1}{n}\frac{v}{r}\prod_{s=1}^{t} p_r^s\right) + r} \\
&= \frac{r}{v}\frac{n}{\left((n-1)\prod_{s=1}^{t-1} p_r^s + P\right)\left(1 - \frac{n-1}{n}\frac{v}{r}\prod_{s=1}^{t} p_r^s\right) + (n-1)\prod_{s=1}^{t} p_r^s} \\
&= \frac{r}{v}\frac{n}{(n-1)\prod_{s=1}^{t-1} p_r^s\left(1 - \frac{n-1}{n}\frac{v}{r}\prod_{s=1}^{t} p_r^s\right) - P\frac{n-1}{n}\frac{v}{r}\prod_{s=1}^{t} p_r^s + P + (n-1)\prod_{s=1}^{t} p_r^s} \\
&= \frac{r}{v}\frac{n}{(n-1)\left(\prod_{s=1}^{t-1} p_r^s - \frac{1}{p_r^t}\prod_{s=1}^{t} p_r^s\right) + P + (n-1)\prod_{s=1}^{t} p_r^s} \\
&= \frac{r}{v}\frac{n}{P + (n-1)\prod_{s=1}^{t} p_r^s}
\end{aligned}$$

By induction $p_r^t = \frac{r}{v}\frac{n}{P+(n-1)\prod_{s=1}^{t-1} p_r^s}$ and with $\lim_{t\to\infty} x^t = 0$, $\lim_{t\to\infty} p_r^t = 1$ and $P = \prod_{s=1}^{\infty} p_r^s$ we have

$$\begin{aligned}
1 &= \lim_{t\to\infty} p_r^t = \lim_{t\to\infty} \frac{r}{v}\frac{n}{P + (n-1)\prod_{s=1}^{t-1} p_r^s} \\
\Rightarrow\ 1 &= \frac{r}{v}\frac{n}{P + (n-1)P} = \frac{r}{v}\frac{1}{P} \Rightarrow P = \frac{r}{v}
\end{aligned}$$

□

In the next lemma we provide an explicit equilibrium expression for effort in a given period t.

Lemma 5 *In the unique equilibrium of Model A with an infinite time horizon each player exerts*

$$x^t = \frac{n-1}{n^2}(v-r)\frac{r}{v(n^{t-1}-1)+r} \quad \textit{in period } t = 1, 2, \ldots.$$

PROOF : With equation (1) and $\sum_{t=1}^{\infty} p^t \prod_{s=1}^{t-1} p_r^s = \frac{1}{n}(1 - \prod_{t=1}^{\infty} p_r^t) = \frac{1}{n}(1 - \frac{r}{v})$ from Lemma 4 we have for period $t = 1$

$$\frac{p^1}{x^1}\left(1 - \frac{1}{n}\left(1 - \frac{r}{v}\right)\right) = \frac{1}{v} \Leftrightarrow x^1 = \frac{n-1}{n^2}(v-r)\frac{r}{v(n^0-1)+r} \ ,$$

$$p_r^1 = \frac{r}{n \cdot x^1 + r} = \frac{r \cdot n}{(n-1)v + r}$$

and with equation (5) we have for period $t = 2$

$$x^2 = x^1\left(1 - \frac{n-1}{n}\frac{v}{r}p_r^1\right) = \frac{n-1}{n^2}(v-r)\frac{r}{v(n^1-1)+r} \ ,$$

$$p_r^2 = \frac{r}{n \cdot x^2 + r} = n\frac{(n-1)v+r}{(n^2-1)v+r}$$

and

$$p_r^1 \cdot p_r^2 = \frac{r \cdot n}{(n-1)v+r} n \frac{(n-1)v+r}{(n^2-1)v+r} = \frac{r \cdot n^2}{(n^2-1)v+r}$$

Suppose now

$$x^t = \frac{n-1}{n^2}(v-r)\frac{r}{v(n^{t-1}-1)+r}$$

and

$$\prod_{s=1}^{t} p_r^s = \frac{r \cdot n^t}{v(n^t-1)+r}$$

for some $t = 1, 2, \ldots$. By equation (5) we then have for period $t+1$

$$\begin{aligned}
x^{t+1} &= \frac{n-1}{n^2}(v-r)\frac{r}{v(n^{t-1}-1)+r}\left(1 - \frac{n-1}{n}\frac{v}{r}\frac{r \cdot n^t}{v(n^t-1)+r}\right) \\
&= \frac{n-1}{n^2}(v-r)\frac{r}{v(n^{t-1}-1)+r}\frac{v(n^{t-1}-1)+r}{v(n^t-1)+r} \\
&= \frac{n-1}{n^2}(v-r)\frac{r}{v(n^t-1)+r}
\end{aligned}$$

and

$$\begin{aligned}
p_r^{t+1} &= \frac{r}{n \cdot x^{t+1} + r} = \frac{r}{n\frac{n-1}{n^2}(v-r)\frac{r}{v(n^t-1)+r} + r} \\
&= \frac{v(n^t-1)+r}{\frac{n-1}{n}(v-r) + v(n^t-1) + r} = n\frac{v(n^t-1)+r}{v(n^{t+1}-1)+r}
\end{aligned}$$

and hereby

$$\prod_{s=1}^{t+1} p_r^s = \frac{r \cdot n^t}{v(n^t-1)+r} n \frac{v(n^t-1)+r}{v(n^{t+1}-1)+r} = \frac{r \cdot n^{t+1}}{v(n^{t+1}-1)+r} = \frac{r}{v-(v-r)n^{-(t+1)}} .$$

The statement of the proposition follows by induction. □

3.3 Revenue Comparison in Model A

The explicit equilibrium characterizations for the finite and infinite time horizon allow us to calculate and compare total equilibrium effort and therefore contest revenue. With a finite time horizon contest revenue is as high as in the benchmark model, or alternatively, in the standard one-stage contest because intertemporal effort substitution does not affect total effort exertion (Corollary 1). The expression for total effort in the infinite case is instead more complex because individual effort exertion over time differs substantially from the benchmark as well as the finite time horizon (Proposition 3). The following result demonstrates, however, that intertemporal effort substitution under the infinite time horizon tends to decrease contest revenue: Except for the case of two contestants combined with a small draw parameter, contest revenue under an infinite horizon is typically lower than under a finite time horizon. We provide an intuitive explanation after stating the formal result.

Proposition 4 *In model A contest revenue is generally higher under a finite than an infinite time horizon except for $n = 2$ and a sufficiently small draw parameter r.*

Proof : In model A with in infinite time horizon total equilibrium effort X^A_∞ is given by the following expression directly derived from Proposition 3:

$$X^A_\infty(n, v, r) := n \sum_{t=1}^{\infty} x^t = \frac{n-1}{n}(v-r) \sum_{t=0}^{\infty} \frac{r}{v(n^t-1)+r}.$$

We are now going to show that for $n = 2$ there exists some $\hat{r} \in (0, v)$ such that $X^A_\infty(n, v, r) > \frac{n-1}{n}v$ for all $r \in (0, \hat{r})$, while for $n > 2$ the inequality $X^A_\infty(n, v, r) < \frac{n-1}{n}v$ holds for all $r > 0$.
Note, that $\lim_{r\to 0} X^A_\infty(n, v, r) = \frac{n-1}{n}v$ and $\lim_{r\to v} X^A_\infty(n, v, r) = 0$. The

derivative of $X_\infty^A(n,v,r)$ with respect to r can be calculated as follows:

$$\begin{aligned}\frac{\partial X_\infty^A(n,v,r)}{\partial r} &= -\frac{n-1}{n}\sum_{t=0}^{\infty}\frac{r}{v(n^t-1)+r}+\frac{n-1}{n}(v-r)\sum_{t=0}^{\infty}\frac{v(n^t-1)}{(v(n^t-1)+r)^2}\\ &= -\frac{n-1}{n}\left(\sum_{t=0}^{\infty}\frac{r^2+2rv(n^t-1)-v^2(n^t-1)}{(v(n^t-1)+r)^2}\right).\\ &= -\frac{n-1}{n}\left(1+\sum_{t=1}^{\infty}\frac{r^2+2rv(n^t-1)-v^2(n^t-1)}{(v(n^t-1)+r)^2}\right).\end{aligned}$$

Taking the limit of r to 0 yields the following expression:

$$\begin{aligned}\left.\frac{\partial X_\infty^A(n,v,r)}{\partial r}\right|_{r\downarrow 0} &= -\frac{n-1}{n}+\frac{n-1}{n}\sum_{t=1}^{\infty}\frac{1}{n^t-1}=\frac{n-1}{n}\left[\sum_{t=1}^{\infty}\left(\frac{1}{n^t-1}\right)-1\right]\\ &= \frac{n-1}{n}\sum_{t=1}^{\infty}\left(\frac{1}{n^t-1}-\frac{1}{2^t}\right).\end{aligned}$$

For $n=2$ we have that $\frac{1}{2^t-1}-\frac{1}{2^t}>0$ for all $t\geq 1$. Hence, for $n=2$ the sign of the derivative is positive close to the lower boundary of r. By continuity of $X_\infty^A(n,v,r)$ in $r>0$ there exists some $\hat{r}\in(0,v)$ such that $X_\infty^A(n,v,r)>\frac{n-1}{n}v$ for all $r\in(0,\hat{r})$.

For $n>2$ we have that $\frac{1}{n^t-1}\leq\frac{1}{2^t}$ with a strict inequality for $t>1$.[11] Therefore the sign of the derivative is negative for $n>2$ close to the lower boundary of r. It remains to be shown that $X_\infty^A(n,v,r)$ is concave in r to deduce that $X_\infty^A(n,v,r)<\frac{n-1}{n}v$ for all $n>2$:

$$\begin{aligned}\frac{\partial^2 X_\infty^A(n,v,r)}{\partial r^2} &= -\frac{n-1}{n}\sum_{t=0}^{\infty}\left(\frac{2v\left(n^t-1\right)+2r}{\left(v\left(n^t-1\right)+r\right)^2}-\frac{2\left(2rv\left(n^t-1\right)-v^2\left(n^t-1\right)+r^2\right)}{\left(v\left(n^t-1\right)+r\right)^3}\right)\\ &= -\frac{n-1}{n}\sum_{t=0}^{\infty}\left(\frac{2v^2n^t\left(n^t-1\right)}{\left(v\left(n^t-1\right)+r\right)^3}\right)<0,\end{aligned}$$

where the inequality holds because all elements in the sum are either zero (for $t=0$) or positive (for $t>0$). □

Proposition 4 implies that the potential for additional revenue generation beyond the benchmark is limited to the two player case, compare also Figure 1 for numerical results. An intuition for this result can be related to the fact

[11] We have $n^1-1\geq 2^1$ for all $n>2$ and $\frac{\partial n^t-1}{\partial t}=n^t\ln(n)>2^t\ln(2)=\frac{\partial 2^t}{\partial t}$ for all $n>2$.

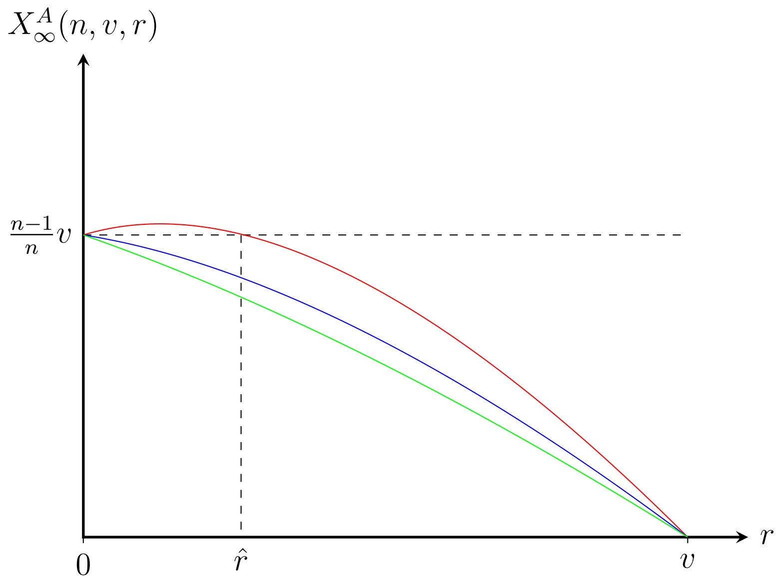

Figure 1: Contest revenue in Model A with infinite time horizon for $n = 2$ (red), $n = 3$ (blue), and $n = 5$ (green)

that exerting effort does not only increase the winning probability in a given period but also reduces the probability that the game continues to the next period. The second mentioned effect can be exploited to induce additional effort beyond the benchmark setup. However, this effect is more pronounced if there are few players present in combination with a low draw parameter.[12]

4 Analysis of Model B

In this section we analyze Model B where the players' decision to exert effort in a specific period is actually made in the respective period and can therefore be conditioned on the fact whether this period has been reached. This implies that in contrast to Model A the expected payoff for a specific period t is period-dependent while future effort costs are probabilistic (in the sense that future effort is only exerted if these periods are actually reached, requiring a draw in all previous periods) and past effort costs are sunk. Denote

[12]If the draw parameter is relatively small then effort exertion is more efficient in reducing the probability of a draw. The same relation holds if the number of players is comparatively low (at least in the first period that is more relevant for revenue generation as it is reached with certainty). Lemma 5 can be used, for instance, to show that x^1 in the infinite time horizon case is larger than x^1 in the finite time horizon case and that this difference is decreasing in n.

$x_i^{-t} = (x_i^1, \ldots, x_i^{t-1}, x_i^{t+1}, \ldots, x_i^T)$ and $x_{-i}^{-t} = (x_1^{-t}, \ldots, x_{i-1}^{-t}, x_{i+1}^{-t}, \ldots, x_n^{-t})$. The expected payoff function for period t can then be expressed as follows:[13]

$$E_t[u_i(x_i^t, x_{-i}^t)|(x_i^{-t}, x_{-i}^{-t})] = \sum_{k=t}^{T}\left(\prod_{s=t}^{k-1} p_r^s\right)(v \cdot p_i^k - x_i^k) - \sum_{k=1}^{t-1} x_i^k. \qquad (6)$$

The dynamic structure of the game suggests subgame-perfection as the appropriate equilibrium concept. Before we characterize the subgame-perfect equilibrium for the finite and infinite time horizon in the next two subsections, we first present a helpful lemma which facilitates our equilibrium characterization. Using the first order conditions for two subsequent periods (as in Section 3.2), we can derive a difference equation that has to be satisfied in any interior equilibrium of the dynamic contest game in Model B.

Lemma 6 *In any interior and symmetric subgame perfect equilibrium of Model B effort exertion in two subsequent periods t and $t+1$ with $t < T-1$ has to satisfy the following equation:*

$$x^{t+1} = \left(\frac{2nr - (n-1)v}{(n+1)r} + \frac{n^2}{(n+1)r}x^t\right)x^t.$$

Proof : The first order condition for an interior subgame perfect equilibrium in period $t < T$ based on (6) can be expressed as follows (exploiting again the fact that $\frac{\partial p_i^t}{\partial x_i^t} = (1-p_i^t)\frac{p_i^t}{x_i^t}$ and $\frac{\partial p_r^t}{\partial x_i^t} = -\frac{(p_r^t)^2}{r}$ for $t < T$):

$$\begin{aligned}
\frac{\partial E_t[u_i(x_i^t, x_{-i}^t)]}{\partial x_i^t} &= \left(v\frac{\partial p_i^t}{\partial x_i^t} - 1\right) + \sum_{k=t+1}^{T}\left(\prod_{s=t}^{k-1} p_r^s\right)(v \cdot p_i^k - x_i^k)\frac{1}{p_r^t}\frac{\partial p_r^t}{\partial x_i^t} = 0 \\
\Leftrightarrow \quad & \frac{p_i^t}{x_i^t}\left(v(1-p_i^t) - \sum_{k=t+1}^{T}\left(\prod_{s=t}^{k-1} p_r^s\right)(v \cdot p_i^k - x_i^k)\right) = 1 \qquad (7) \\
\Leftrightarrow \quad & v(1-p_i^t) - p_r^t(v \cdot p_i^{t+1} - x_i^{t+1}) - p_r^t \sum_{k=t+2}^{T}\left(\prod_{s=t+1}^{k-1} p_r^s\right)(v \cdot p_i^k - x_i^k) = \frac{x_i^t}{p_i^t}. \qquad (8)
\end{aligned}$$

[13] The conventions $\prod_{s=t}^{t-1} p_r^s = 1$ and $\sum_{k=t+1}^{t} f(\cdot) = 0$ from the previous subsection are also applied in this subsection.

Based on (7) a similar equation can be derived for period $t+1 < T$:

$$\frac{p_i^{t+1}}{x_i^{t+1}}\left(v(1-p_i^{t+1}) - \sum_{k=t+2}^{T}\left(\prod_{s=t+1}^{k-1} p_r^s\right)(v \cdot p_i^k - x_i^k)\right) = 1 \tag{9}$$

$$\Leftrightarrow \quad \sum_{k=t+2}^{T}\left(\prod_{s=t+1}^{k-1} p_r^s\right)(v \cdot p_i^k - x_i^k) = v(1-p_i^{t+1}) - \frac{x_i^{t+1}}{p_i^{t+1}} \tag{10}$$

Substituting equation (10) into equation (8) and assuming symmetry yields:

$$\begin{aligned} & v(1-p_i^t-p_r^t) + p_r^t x_i^{t+1} + p_r^t \frac{x_i^{t+1}}{p_i^{t+1}} = \frac{x_i^t}{p_i^t} \\ \Leftrightarrow \quad & v(n-1)x^t + rx^{t+1} + r(nx^{t+1}+r) = (nx^t+r)^2 \\ \Leftrightarrow \quad & x^{t+1} = \left(\frac{2nr-(n-1)v}{(n+1)r} + \frac{n^2}{(n+1)r}x^t\right)x^t. \end{aligned} \tag{11}$$

□

Note that the resulting difference equation is of first order (in contrast to the difference equation derived for Model A); however, it is still non-linear and cannot be expressed typically in closed form.[14] Nevertheless, it is instrumental in characterizing the subgame-perfect equilibria for the case of finite and infinite time horizon.

4.1 Finite Time Horizon

Under a finite time horizon the subgame-perfect Nash equilibrium of this extensive form game can be characterized by backward induction. Hence, we start the equilibrium analysis with the last period. If the last period is actually reached, the game becomes a standard one-stage Tullock lottery contest because there is no draw by definition. Hence, each player chooses $x^T = \frac{n-1}{n^2}v$ with $p^T = \frac{1}{n}$ in the unique equilibrium of any subgame starting in period T.

Analyzing the equilibrium effort in period $T-1$ requires to take into account the possibility of non-interior equilibria because depending on the likelihood of a draw players might prefer to wait and exert effort only in the

[14]For the specific parameter value $r = \frac{v(n-1)}{2n}$ we are able to find a closed form expression which is $x^t = \frac{v}{2}\frac{n^2-1}{n}\left(\frac{\sqrt{1+\frac{2}{n}}}{n(n+1)}\right)^{2^{t+1-T}}$.

last period. Hence, we can slightly modify (7) to allow for the case of zero equilibrium effort in period $T-1$ and apply our results for the last period which results in the following inequality, where we used symmetry to derive the last inequality.

$$\begin{aligned} \frac{p_r^{T-1}}{r}\left(v(1-p_i^{T-1})-(v\cdot p_i^T-x_i^T)p_r^{T-1}\right) &\leq 1 \\ \Leftrightarrow \quad v(n-1)\left(x^{T-1}+(n+1)\frac{r}{n^2}\right) &\leq (n\cdot x^{T-1}+r)^2 \end{aligned}$$

If $x^{T-1}>0$, then the last inequality must be satisfied by equality, which implies

$$x^{T-1}=\frac{\sqrt{v(n-1)}}{2n^2}\cdot\left(\sqrt{v(n-1)+4r}+\sqrt{v(n-1)}\right)-\frac{r}{n}\ ,$$

which is positive only if $r<\frac{n^2-1}{n^2}v$. If $r\geq\frac{n^2-1}{n^2}v$, the inequality is strict for any $x^{T-1}>0$. In this case $x^{T-1}=0$ must hold in equilibrium. Before we characterize the interior equilibrium for $r<\frac{n^2-1}{n^2}v$, we establish that $x^t=0$ must hold for all $t<T$ in any symmetric subgame perfect equilibrium for $r\geq\frac{n^2-1}{n^2}v$.

Lemma 7 *Let $r\geq\frac{n^2-1}{n^2}v$ and consider Model B with $T<\infty$. If the equilibrium is symmetric, then:*

$$x_i^t=0 \textit{ for all } t<T \textit{ and } x_i^T=\frac{n-1}{n^2}v \textit{ for all } i=1,\ldots,n.$$

Proof : For the last period T standard arguments imply that $x_i^T=\frac{n-1}{n^2}v$ and $v\cdot p_i^T-x_i^T=v\frac{1}{n^2}$ for all $i=1,\ldots,n$. For any period $t<T$ the first derivative of player i's utility function as defined in equation (6) with respect to x_i^t is given by:

$$\frac{\partial u_i^t}{\partial x_i^t}=v\frac{p_r^t}{r}\left(1-p_i^t\right)-1-\frac{\left(p_r^t\right)^2}{r}\sum_{k=t+1}^{T}\left(\prod_{s=t+1}^{k-1}p_r^s\right)\left(vp_i^k-x_i^k\right).$$

To show $x_i^t=0$ for all $t<T$ we proceed by (backward) induction.
We show firstly that if $\sum_{k=t+1}^T\left(\prod_{s=t+1}^{k-1}p_r^s\right)\left(v\cdot p_i^k-x_i^k\right)=v\frac{1}{n^2}$ for some $t<T$ then the first order condition implies $p_i^t=0$.
With $\sum_{j=1}^n p_j^t+p_r^t=1$ and symmetry we have $1-p_i^t=\frac{n-1+p_r^t}{n}$. Therefore

$$\frac{\partial u_i^t}{\partial x_i^t}=v\frac{p_r^t}{r}\frac{n-1+p_r^t}{n}-1-\frac{\left(p_r^t\right)^2}{r}v\frac{1}{n^2}=\frac{v}{r}\frac{n^2-1}{n^2}p_r^t\frac{n+p_r^t}{n+1}-1.$$

In equilibrium we must have $\frac{\partial u_i^t}{\partial x_i^t} \leq 0$, with equality if $x_i^t > 0$. The derivative is strictly increasing in p_r^t and negative for $p_r^t < 1$ because $r \geq \frac{n^2-1}{n^2}v$. This implies that $x_i^t = 0$ for all $i = 1, \ldots, n$.
Hence, if for some $t < T$ we have $\sum_{k=t+1}^{T} \left(\prod_{s=t+1}^{k-1} p_r^s\right)\left(v \cdot p_i^k - x_i^k\right) = v\frac{1}{n^2}$ then with $x_i^t = 0$ for all i and $p_r^t = 1$ we have $\sum_{k=t}^{T} \left(\prod_{s=t}^{k-1} p_r^s\right)\left(v \cdot p_i^k - x_i^k\right) = v\frac{1}{n^2}$. As $v \cdot p_i^T - x_i^T = v\frac{1}{n^2}$, we have by induction $x_i^t = 0$ for all $t < T$ and $i = 1, \ldots, n$. □

For $r < \frac{n^2-1}{n^2}v$ we continue with the backward induction process for period $T-2$ and determine x_i^{T-2} in the same way or, alternatively, by inverting the difference equation from Lemma 6 and using the equilibrium results from period $T-1$. Unfortunately, the resulting expression contains nested roots in a convoluted way that cannot be simplified further. Hence, it is not possible to obtain a simple closed form expression for equilibrium effort in each period. However, using the difference equation from Lemma 6 the following qualitative equilibrium properties can be derived.

Proposition 5 *The subgame-perfect equilibrium of Model B with finite time horizon is (i) unique and symmetric. Equilibrium effort is (ii) bounded in each period and (iii) (weakly) decreasing over time except for the last period. Moreover, (iv) equilibrium behavior in the last periods is not affected by the total number of periods, while (v) effort in the first period converges (if positive) to the upper bound.*

We prove each statement (i) – (v) of the proposition in a separate lemma.

Lemma 8 *The subgame-perfect equilibrium is unique and symmetric.*

Proof : As the subgame-perfect equilibrium can be determined by backward induction, we can establish uniqueness and symmetry by analyzing the equilibrium strategy in each subgame. In the last period the game is equivalent to a one-stage standard lottery contest which has a unique and symmetric equilibrium. In the penultimate period the (reduced) game is then equivalent to a one-stage lottery contest with draws, which has a unique equilibrium, see Jensen (2016) for an existence and uniqueness proof for a general class of contest games with draws which contains our stage game as a special case. As players are homogeneous and use symmetric strategies in the last period, the symmetry property can be established easily for $T-1$ by using the respective equilibrium characterization. The same argument can then be applied recursively in all other periods. □

Lemma 9 *Equilibrium effort is bounded.*

Proof : For period T the equilibrium effort is given by $x^T = \frac{n-1}{n^2}v$, for period $T-1$ the equilibrium effort is given by

$$x^{T-1} = \begin{cases} \frac{\sqrt{v(n-1)}}{2n^2}\left(\sqrt{v(n-1)+4r}+\sqrt{v(n-1)}\right) - \frac{r}{n} & \text{if } r < \frac{n^2-1}{n^2}v, \\ 0 & \text{otherwise.} \end{cases}$$

For all periods $t < T-1$ and $r < \frac{n^2-1}{n^2}v$, the equilibrium effort is characterized through the difference equation from Lemma 6, which can be expressed as

$$x^{t+1} = \left(A + B \cdot x^t\right) x^t \ ,$$

with $A = \frac{2n \cdot r - (n-1)v}{(n+1)r} \in \mathbb{R}$ and $B = \frac{n^2}{(n+1)r} \in \mathbb{R}_{++}$. We then have $x^{t+1} = x^t \Leftrightarrow x^t = 0 \vee x^t = \frac{1-A}{B} = \frac{n-1}{n^2}(v-r) =: \bar{x}$. Further, we can invert the difference equation to obtain[15]

$$x^t = -\frac{A}{2B} + \sqrt{\left(\frac{A}{2B}\right)^2 + \frac{1}{B}x^{t+1}}.$$

Obviously, $x^{t+1} > 0 \Rightarrow x^t > 0$. Suppose now that $x^{t+1} < \frac{1-A}{B}$. Then

$$x^t < -\frac{A}{2B} + \sqrt{\left(\frac{A}{2B}\right)^2 + \frac{1}{B}\frac{1-A}{B}} = -\frac{A}{2B} + \sqrt{\frac{4-4A+A^2}{4B^2}} = \frac{1-A}{B}$$

It is straight forward to show that $x^{T-1} < \frac{n-1}{n^2}(v-r)$, hence we have $x^t < \bar{x}$ for $t = 1, \ldots, T-1$. □

Lemma 10 *Equilibrium effort is (weakly) decreasing for* $t = 1, \ldots, T-1$.

Proof : The proof consists of two parts. In the first we show that $x^t > 0$ and $x^{t+1} > 0$ imply $x^t > x^{t+1}$. In the second part we show that $x^t = 0$ implies $x^{t+1} = 0$.
Part 1: Suppose that $x^{t+1} > 0$ and $x^t > 0$ for some $t < T-1$. Then

[15] We can ignore the second solution of this quadratic equation because $x^t = -\frac{A}{2B} - \sqrt{\left(\frac{A}{2B}\right)^2 + \frac{1}{B}x^{t+1}} < 0$ for any $A \in \mathbb{R}$.

$x^{t+1} = (A + Bx^t)x^t$. We have $x^t > x^{t+1} \Leftrightarrow A + Bx^t < 1 \Leftrightarrow x^t < \frac{1-A}{B} = \bar{x}$, which was shown in Lemma 9.
Part 2: Suppose now that in a symmetric SPE $x^t = 0$ holds for some $t < T-1$. Then the first order condition from equation (8) holds as inequality and can be simplified as follows:

$$v(1 - p^{t+1}) - r + x^{t+1} \leq \sum_{k=t+2}^{T} \left(\prod_{s=t+1}^{k-1} p_r^s \right) (v \cdot p^k - x^k) \ .$$

Suppose now (by contradiction) that $x^{t+1} > 0$. This implies that

$$v(1 - p^{t+1}) - r - nx^{t+1} < \sum_{k=t+2}^{T} \left(\prod_{s=t+1}^{k-1} p_r^s \right) (v \cdot p^k - x^k)$$

which is equivalent to

$$\frac{p^{t+1}}{x^{t+1}} \left(v(1 - p^{t+1}) - \sum_{k=t+2}^{T} \left(\prod_{s=t+1}^{k-1} p_r^s \right) (v \cdot p^k - x^k) \right) - 1 < 0 \ .$$

However, this inequality is in contradiction to equation (9). Therefore, $x^{t+1} = 0$ has to hold. □

Lemma 11 *Equilibrium effort in the last periods is not affected by the total number of periods: Assume that $T' > T$ then $x_{T-k} = x_{T'-k}$ for all $k \leq T$.*

Proof : Equilibrium effort for periods T and $T - 1$ has been already derived in explicit form. By inspection they do not depend on T. The backward induction method implies that the total number of periods T will also not affect effort exertion in previous periods. □

Lemma 12 *Equilibrium effort in the first period converges to the upper bound (whenever $x^1 > 0$): $\lim_{T\to\infty} x^1 = \bar{x}$.*

Proof : Lemma 10 implies that equilibrium effort is decreasing over time (whenever $x^1 > 0$), while Lemma 11 implies that equilibrium effort is fixed for the last periods irrespectively of the total number of periods. Hence, effort exertion in the first period must converge to the upper bound $\bar{x}$ for T large. □

The indirect equilibrium characterization in Proposition 5 does not allow for an explicit functional form expression for expected contest revenue.[16] However, it allows us to establish a lower bound for expected contest revenue which coincides with revenue from the benchmark model.

Proposition 6 *Contest revenue in Model B with a finite time horizon is bounded below and higher than in the benchmark model if* $r < \frac{n^2-1}{n^2}v$*:*

$$X_T^B := n\sum_{t=1}^{T} x^t \prod_{s=1}^{t-1} p_r^s > \frac{n-1}{n}v \text{ for all } T \geq 2.$$

Moreover, contest revenue is increasing in T *if* $r < \frac{n^2-1}{n^2}v$.
If $r \geq \frac{n^2-1}{n^2}v$ *then contest revenue is the same as in the benchmark model.*

PROOF : For $r \geq \frac{n^2-1}{n^2}v$ Lemma 7 in combination with Proposition 5 implies that $X_T^B = \frac{n-1}{n}v$. For $r < \frac{n^2-1}{n^2}v$ the proof consists of two parts. In the first part we show that contest revenue is bounded below by the revenue from the benchmark model, in the second part we establish that contest revenue is increasing in T.
Part 1: Define $E_t[X] := \sum_{k=t}^{T} x^k \prod_{s=t}^{k-1} p_r^s = x^t + p_r^t \cdot E_{t+1}[X]$ and note that $E_1[X] = \frac{X_T^B}{n}$ and that $E_T[X] = x^T = \frac{n-1}{n^2}v$. We prove the claim $X_T^B > \frac{n-1}{n}v$ by showing that $E_t[X] > E_{t+1}[X]$ for all $t = 1, \ldots, T-1$ which implies that $E_1[X] = \frac{X_T^B}{n} > E_T[X] = \frac{n-1}{n^2}v$. The proof proceeds by induction. Assume first that $E_t[X] > E_{t+1}[X] \Leftrightarrow x^t + p_r^t \cdot E_{t+1}[X] > E_{t+1}[X] \Leftrightarrow x^t > (1-p_r^t) \cdot E_{t+1}[X] \Leftrightarrow x^t > n \cdot p^t \cdot E_{t+1}[X] \Leftrightarrow n \cdot x^t + r > n \cdot E_{t+1}[X] \Leftrightarrow\ x^t + \frac{r}{n} > E_{t+1}[X]$.
We now establish that $E_{T-1}[X] > E_T[X]$. For $t = T-1$ and $r < \frac{n^2-1}{n^2}v$ we have $x^{T-1} = \frac{\sqrt{v(n-1)}}{2n^2}\left(\sqrt{v(n-1)+4r} + \sqrt{v(n-1)}\right) - \frac{r}{n}$ and $E_T[X] = x^T = \frac{n-1}{n^2}v$. Therefore $E_{T-1}[X] > E_T[X] \Leftrightarrow \frac{\sqrt{v(n-1)}}{2n^2}\left(\sqrt{v(n-1)+4r} + \sqrt{v(n-1)}\right) - \frac{r}{n} + \frac{r}{n} > \frac{n-1}{n^2}v \Leftrightarrow r > 0$. In the last step we prove that if $E_t[X] > E_{t+1}[X]$ holds for some $t < T$ then $E_{t-1}[X] > E_t[X]$ holds as well. Suppose now that $E_t[X] > E_{t+1}[X]$ for some $t < T$. From Proposition 5 we know that $x^{t-1} > x^t$ for any $t < T$. Therefore $n\left(x^{t-1} - x^t + \frac{r}{n}\right) \cdot E_{t+1}[X] > r \cdot E_{t+1}[X]$. As $x^t + \frac{r}{n} > E_{t+1}[X]$, we have $n\left(x^{t-1} - x^t + \frac{r}{n}\right)\left(x^t + \frac{r}{n}\right) >$

[16] In Model B contest revenue is interpreted in expected terms because effort is only exerted if the expected period has been reached, i.e., there must have been draws in all previous periods.

$r \cdot E_{t+1}[X]$. With $E_t[X] = x^t + p_r^t \cdot E_{t+1}[X] \Leftrightarrow E_{t+1}[X] = \frac{E_t[X]-x^t}{p_r^t}$, we have $n\left(x^{t-1} - x^t + \frac{r}{n}\right)\left(x^t + \frac{r}{n}\right) > r\frac{E_t[X]-x^t}{p_r^t} = n(x^t + \frac{r}{n})(E_t[X] - x^t)$ which is equivalent to $x^{t-1} + \frac{r}{n} > E_t[X]$, implying that $E_{t-1}[X] > E_t[X]$. Hence, we proved that $E_t[X] > E_{t+1}[X]$ for all $t = 1, \ldots, T-1$.

Part 2: To consider the effect of an increment of the finite time horizon from T to $T+1$, we include T in the definition of the expected equilibrium expenditure: $E_{t,T}[X]$. Let $\mathbf{x}$ be the equilibrium vector under time horizon T and $\tilde{\mathbf{x}}$ the equilibrium vector under time horizon $T+1$. From Lemma 11, $\mathbf{x}^t = \tilde{\mathbf{x}}^{t+1}$ must hold for $t = 1, \ldots, T$. Accordingly, let X be associated with T and $\tilde{X}$ with $T+1$. By the previous results from part 1 we have $E_{1,T+1}[\tilde{X}] > E_{2,T+1}[\tilde{X}] = E_{1,T}[X]$, implying that $\tilde{X} > X$. □

4.2 Infinite Time Horizon

Under an infinite time horizon the equilibrium cannot be characterized by backward induction. Instead, we rely on the notion of a stationary equilibrium, where equilibrium effort in a specific round is history independent. Applying this equilibrium concept in our framework is justified because the specific situation that a player faces in period t is basically strategically equivalent to the situation in period $t+1$: Comparing the two expected payoff functions for period t and $t+1$ implies that a player faces a potentially infinite amount of future periods, where she nevertheless can condition future effort decisions on the fact that the respective period has been reached. The only difference between the two situations relates to the sunk effort costs in period t given that period $t+1$ has been reached, that are, however, irrelevant for the effort decision in period $t+1$. Hence, a stationary equilibrium must imply that effort exertion is constant over time. As the difference equation from Lemma 6 holds for a finite and infinite time horizon, we can identify the stationary equilibrium as the positive stationary point of the difference equation.

Proposition 7 *There is a unique stationary equilibrium in Model B with infinite time horizon, where* $x^t = \bar{x} = (v-r)\frac{n-1}{n^2}$ *for all* t.

The following implications regarding winning probabilities and contest revenue follow from this closed form equilibrium characterization.

Corollary 2 *For Model B with infinite time horizon the following statements hold: (i) In the stationary equilibrium one of the players wins the contest with* *probability one (i.e., there is no perpetual draw). (ii) Total effort exertion in Model B with infinite time horizon is higher than with a finite time horizon.*

Proof : The proof of (i) is as follows. Note that $p_r^t = p_r = \frac{r}{n \cdot \bar{x} + r} = \frac{n \cdot r}{(n-1)v+r} < 1$ for all t in the stationary equilibrium. The probability that anyone of the players wins the overall contest game can then be calculated as follows:

$$n \sum_{t=1}^{\infty} p^t \prod_{s=1}^{t-1} p_r^s = 1 - \prod_{t=1}^{\infty} p_r^t = 1 - \lim_{t \to \infty} \left(\frac{r}{n \cdot \bar{x} + r} \right)^t = 1.$$

To prove (ii) it is sufficient to recall that per period effort in Model B with finite time horizon is bounded above by $\bar{x}$ in each period $t < T$ (Lemma 9) and is decreasing (Lemma 10), whereas contest revenue is increasing in T (Proposition 6).[17] Per period effort in Model B with infinite time horizon is, in contrast, equal to the upper bound in each period (Proposition 7). This directly implies that $X_\infty^B > X_T^B$. □

5 Revenue Comparisons

The equilibrium analysis allowed us to derive contest revenue either explicitly or to provide appropriate bounds for all considered variants in our framework. We were also able to compare contest revenue within model A and B, respectively. The following result complements these comparisons based on direct revenue comparison between Model A and B, which leads to a revenue ranking among all variants in our framework. While this ranking depends on the underlying parameter values, revenue-dominance of Model B with infinite horizon can be established independently of the underlying parameter values and is therefore robust. This result is specifically important for a contest organiser who has substantial discretionary power with respect to the design of the contest (including the time horizon dimension and the effort timing dimension) and can therefore implement the preferred variant directly. In this case implementing Model B with an infinite time horizon induces the highest contest revenue among all four variants irrespective of the distribution of parameter values (r, v, n).

[17] These three properties imply that effort exertion in the first periods (where $x^t < \bar{x}$) contributes more to contest revenue than effort exertion in the last period (where $x^T > \bar{x}$).

Proposition 8 *Contest revenue in Model B with infinite time horizon is higher than in all other considered variants.*

Depending on parameter values for (r, n, v), the following revenue ranking holds:

(i) For $n > 2$:

$$X^B_\infty > X^B_T > X^A_T > X^A_\infty.$$

(ii) For $n = 2$ with r sufficiently small and T large:

$$X^B_\infty > X^B_T > X^A_\infty > X^A_T.$$

(iii) For $n = 2$ with r and T sufficiently small:

$$X^B_\infty > X^A_\infty > X^B_T > X^A_T.$$

Proof : To prove revenue-dominance of Model B with an infinite time horizon, we calculate expected contest revenue in this framework using the explicit equilibrium characterization from Proposition 7 which simplifies as follows:

$$X^B_\infty := n \sum_{t=1}^{\infty} \bar{x} \prod_{s=1}^{t-1} p_r = n \cdot \bar{x} \sum_{t=0}^{\infty} \left(\frac{r}{n\bar{x} + r} \right)^t = \frac{n-1}{n} v + \frac{r}{n}.$$

Note firstly, that $X^B_\infty > \frac{n-1}{n} v$, which directly implies that $X^B_\infty > X^A_T$. Secondly, we already established that per period effort in Model B with finite time horizon is bounded above by $\bar{x}$ in each period $t < T$ (Lemma 9) and is decreasing (Lemma 10), whereas contest revenue is increasing in T and bounded below by X^A_T (Proposition 6).[18] Per period effort in Model B with infinite time horizon is, in contrast, equal to the upper bound in each period (Proposition 7). This directly implies that $X^B_\infty > X^B_T > X^A_T$. It remains to establish, thirdly, the dominance relation between contest revenue in Model A and B with infinite horizon. Proposition 4 implies that except for $n = 2$ and r sufficiently low, $X^A_\infty < \frac{n-1}{n} v$ holds. Hence, for $n > 2$ combining these pairwise relations leads to statement (i).

We now focus on the case $n = 2$ and prove firstly that $X^B_\infty > X^A_\infty$ also extends to this specific case. We then proof the remaining statements (ii) and (iii). For $n = 2$ the inequality $X^B_\infty > X^A_\infty$ simplifies to

$$\frac{v + r}{2} > \frac{v - r}{2} \sum_{t=0}^{\infty} \frac{r}{v(2^t - 1) + r}. \tag{12}$$

[18]These three properties imply that effort exertion in the first periods (where $x^t < \bar{x}$) contributes more to contest revenue than effort exertion in the last period (where $x^T > \bar{x}$).

We now construct an upper bound for the expression on the right-hand side of (12) and show subsequently that the left-hand side is larger than this upper bound. In order to do this we work with the ratio of two subsequent elements of the sum on the right-hand side of (12) which can be expressed as follows:

$$\frac{\frac{r}{v(2^{t+1}-1)+r}}{\frac{r}{v(2^t-1)+r}} = \frac{v(2^t-1)+r}{v(2^{t+1}-1)+r}.$$

This expression is increasing in t because $\frac{\partial \frac{v(2^t-1)+r}{v(2^{t+1}-1)+r}}{\partial t} = \frac{2^t \ln(2)(v-r)}{(v(2^{t+1}-1)+r)^2} > 0$. As $\lim_{t\to\infty} \frac{v(2^t-1)+r}{v(2^{t+1}-1)+r} = \frac{1}{2}$, the ratio of two subsequent elements of the sum is bounded above by $\frac{v(2^t-1)+r}{v(2^{t+1}-1)+r} < \frac{1}{2}$, which implies that $\frac{r}{v(2^{t+1}-1)+r} < \frac{1}{2}\frac{r}{v(2^t-1)+r}$. We use this insight to construct an upper bound for the right-hand side of (12):

$$\begin{aligned} \frac{v-r}{2}\sum_{t=0}^{\infty}\frac{r}{v(2^t-1)+r} &= \frac{v-r}{2}\left(1+\frac{r}{v+r}+\sum_{t=2}^{\infty}\frac{r}{v(2^t-1)+r}\right) \\ &< \frac{v-r}{2}\left(1+\frac{r}{v+r}+\sum_{t=1}^{\infty}\left(\frac{1}{2}\right)^t\frac{r}{v+r}\right) \\ &= \frac{v-r}{2}\left(1+\frac{r}{v+r}+\frac{r}{v+r}\right) = \frac{v-r}{2}\frac{v+3r}{v+r}. \end{aligned}$$

It is then straight forward to demonstrate that contest revenue in Model B with infinite horizon is higher than this upper bound: $X_\infty^B > \frac{v-r}{2}\frac{v+3r}{v+r} \Leftrightarrow \frac{v+r}{2} > \frac{v^2+2rv-3r^2}{v+r} \Leftrightarrow 4r^2 > 0$. Hence, $X_\infty^B > \frac{v-r}{2}\frac{v+3r}{v+r} > X_\infty^A$, which establishes revenue-dominance of Model B with infinite time horizon.

It remains to prove statement (ii) and (iii). We firstly note that $\lim_{T\to\infty} X_T^B = X_\infty^B$ which implies that for T sufficiently large $X_T^B > X_\infty^A$ holds. In combination with the revenue relations just established, this leads to the revenue ranking in (ii). We secondly note that $\lim_{T\to 1} X_T^B = X_T^A$. Hence, for T sufficiently small $X_\infty^A > X_T^B$, which leads to the revenue ranking in (iii). □

Revenue-dominance of Model B with an infinite time horizon can be attributed to two factors: Firstly, exerting effort in Model B versus A is more advantageous for players because they face less risk that effort is wasted as they can condition on whether a specific period has been reached. Secondly, the absence of a final period in the infinite horizon case implies that players do not have an incentive to wait for the last period which would be attractive because there is no draw. Hence, they have no incentive to reallocate effort from earlier to (more risky) later periods and instead exert the same level of effort in each period. The combination of these two factors implies that contest revenue in Model B with an infinite time horizon is higher than in the other three variants.

6 Concluding Remarks

We analyze a simple but comprehensive model of dynamic contests, where contests are repeated if a non-decisive outcome (none of the contestants wins in a specific round) is realised. The framework is simple because it is based on well-established functional forms for this type of contest with draws, which facilitates equilibrium characterisation in a highly complex dynamic setup. It is comprehensive in the sense that we consider different assumptions regarding the time horizon (finite and infinite) as well as the timing of effort decisions (ex-ante and per-period).

Our analysis demonstrates that the dynamic structure of the contest game has implications for equilibrium behavior of the contestants in the sense of intertemporal effort substitution; that is, contestants shift some of their contest effort investments into later periods, where the details depend on the implemented variant as well as the underlying parameter values. Using bounds or even explicit formulae for the resulting contest revenue in each setup, we can demonstrate revenue-dominance of one of our variants and identify factors that drive intertemporal effort substitution. These results are of specific interest for revenue-maximising contest organisers that have sufficient degrees of freedom in designing appropriate contest structures.

Our framework should be interpreted as a first attempt to model the dynamic structure of repeated contests with draws, allowing for several generalizations of interest along different lines, for instance, more general contest success functions or the inclusion of discounting. We conjecture that our results should be at least qualitatively robust for the two mentioned extensions. However, it is not obvious whether robustness is also maintained with respect to more fundamental modifications, for instance, heterogeneity of contestants, risk-aversion, or using alternative functional forms to model draw probabilities (comp. Vesperoni and Yildizparlak (2019) for such an alternative). While the analysis of those extensions goes beyond the scope of this paper, we intend to address these issues in our future research.

Appendix: Additional Results and Proofs

Proposition 9 *In model A the payoff function $E[u_i(\mathbf{x}_i, \mathbf{x}_{-i})]$ is strictly concave in $\mathbf{x}_i$ for all $\mathbf{x}_i \in \mathbb{R}_+^T$ and $\mathbf{x}_{-i} \in \mathbb{R}_+^{(n-1)\cdot(T-1)} \times (\mathbb{R}^{n-1} \setminus \{\mathbf{0}\})$.*

PROOF : The second derivatives of $E[u_i(\mathbf{x}_i, \mathbf{x}_{-i})]$ with respect to x_i^l, $l \leq t$ are given by

$$\begin{aligned} \frac{\partial^2 E[u_i(\mathbf{x_i}, \mathbf{x_{-i}})]}{\partial x_i^t \partial x_i^l} &= -\frac{p_i^l}{x_i^l}\left(\frac{\partial E[u_i(\mathbf{x_i}, \mathbf{x_{-i}})]}{\partial x_i^t} + 1\right) \text{ for } l < t \\ \frac{\partial^2 E[u_i(\mathbf{x_i}, \mathbf{x_{-i}})]}{(\partial x_i^t)^2} &= -2\frac{p_i^t}{x_i^t}\left(\frac{\partial E[u_i(\mathbf{x_i}, \mathbf{x_{-i}})]}{\partial x_i^t} + 1\right) . \end{aligned}$$

Denote the Hessian matrix of $E[u_i(\mathbf{x}_i, \mathbf{x}_{-i})]$ with respect to $\mathbf{x}_i$ by $H(\mathbf{x}_i, \mathbf{x}_{-i})$. Define $\phi_t = \frac{p_i^t}{x_i^t}$ and $f_t = \frac{\partial E[u_i(\mathbf{x_i}, \mathbf{x_{-i}})]}{\partial x_i^t} + 1$ for $t = 1, \ldots, T$ and define the $T \times T$ matrices F and Φ as

$$F = \begin{bmatrix} f_1 & 0 & \ldots & 0 \\ 0 & f_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & f_T \end{bmatrix} \text{ and } \Phi = \begin{bmatrix} 2\phi_1 & \phi_2 & \ldots & \phi_T \\ \phi_1 & 2\phi_2 & \ldots & \phi_T \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1 & \phi_2 & \ldots & 2\phi_T \end{bmatrix} .$$

We then have

$$H_{tl}(\mathbf{x}_i, \mathbf{x}_{-i}) = -\phi_l \cdot f_t = -\frac{p_i^l}{x_i^l}\left(\frac{\partial E[u_i(\mathbf{x_i}, \mathbf{x_{-i}})]}{\partial x_i^t} + 1\right) = \frac{\partial^2 E[u_i(\mathbf{x_i}, \mathbf{x_{-i}})]}{\partial x_i^t \partial x_i^l}$$

for $l < t \leq T$ and, as the second derivatives are continuous, $H_{lt}^k(\mathbf{x}_i, \mathbf{x}_{-i}) = H_{tl}^k(\mathbf{x}_i, \mathbf{x}_{-i})$ for $l < t \leq T$. Further,

$$H_{tt}(\mathbf{x}_i, \mathbf{x}_{-i}) = -2\phi_t f_t = -2\frac{p_i^t}{x_i^t}\left(\frac{\partial E[u_i(\mathbf{x_i}, \mathbf{x_{-i}})]}{\partial x_i^t} + 1\right)$$

for $t \leq T$. Therefore the lower triangle and the diagonal of the Hessian H correspond to the lower triangle and diagonal of $-F \cdot \Phi$ and the upper triangle is given by symmetry of H. Note that $\phi_t \cdot f_l \neq \phi_l \cdot f_t$. The subsequent Lemma 13 shows that $f_t = \frac{\partial E[u_i(\mathbf{x_i}, \mathbf{x_{-i}})]}{\partial x_i^t} + 1 > 0$ for all $t = 1, \ldots, T$. Therefore all entries of H are negative. Hence, for any vector $\mathbf{x}_i \in \mathbb{R}_+^T$, $\mathbf{x_i} \neq \mathbf{0}$, we have $(\mathbf{x}_i)' H \mathbf{x}_i = \sum_{l=1}^T \sum_{t=1}^T x_i^l H_{lt} x_i^t < 0$. Therefore H is negative definite and therefore $E[u_i(\mathbf{x}_i, \mathbf{x}_{-i})]$ is strictly concave in $\mathbf{x}_i$ on $\mathbb{R}_+^T$. □

Lemma 13
$\frac{\partial E[u_i(\mathbf{x_i},\mathbf{x_{-i}})]}{\partial x_i^t} > -1$ *for all* $\boldsymbol{x}_i \in \mathbb{R}_+^T$ *and* $\boldsymbol{x}_{-i} \in \mathbb{R}_+^{(n-1)\cdot(T-1)} \times (\mathbb{R}^{n-1} \setminus \{\boldsymbol{0}\})$.

PROOF : The first derivative of the expected utility function is given by

$$\frac{\partial E[u_i(\mathbf{x_i},\mathbf{x_{-i}})]}{\partial x_i^t} = v\frac{p_i^t}{x_i^t}\left(\prod_{s=1}^{t-1} p_r^s\right)\left(1-\sum_{k=t}^{T} p_i^k \prod_{s=t}^{k-1} p_r^s\right) - 1 \ ,$$

where $\sum_{k=t}^{T} p_i^k \prod_{s=t}^{k-1} p_r^s$ is the probability of winning the prize in period t (given that it has been reached) or in any other future periods. Since $p_r^s > 0$ for all periods $s < T$ and since $\mathbf{x}_{-i}^T \neq \mathbf{0}$, this probability is strictly smaller than 1. As $\frac{p_i^t}{x_i^t} > 0$ for all periods $t = 1, \ldots, T$, the statement of the Lemma holds. □

Lemma 14 *Consider Model A. If* $r \geq v$, *then* $x_i^t = 0$ *for* $t = 1, \ldots, T-1$ *is strictly dominant for any player* $i = 1, \ldots, n$. *If* $T = \infty$, *this is the unique Nash equilibrium. If* $T < \infty$, *then* $x_i^t = 0 \ \forall \ t = 1, \ldots, T-1$ *and* $x_i^T = \frac{n-1}{n^2}v$ *for* $i = 1, \ldots, n$ *is the unique Nash equilibrium.*

PROOF : Suppose $r \geq v$ and consider $T \in \mathbb{N} \cup \{\infty\}$. Fix any $\{\mathbf{x}_j\}_{j=1}^n$ with $\mathbf{x}_j = \left\{x_j^t\right\}_{t=1}^T \in \mathbb{R}_+^T$ for $j = 1, \ldots, n$ with $x_i^t > 0$ for some $i = 1, \ldots, n$ and $t < T$. As $\prod_{s=1}^{t-1} p_r^s \leq 1$ and $p_i^t = \frac{x_i^t}{\sum_{j=1}^n x_j^t + r} < \frac{x_i^t}{r}$ we have $v \cdot p_i^t \cdot \prod_{s=1}^{t-1} p_r^s - x_i^t \leq v \cdot p_i^t - x_i^t < v\frac{x_i^t}{r} - x_i^t \leq 0$. Therefore we have

$$E[u_i(\mathbf{x_i},\mathbf{x_{-i}})] = \sum_{t=1}^{T} \underbrace{\left(v \cdot p_i^t \prod_{s=1}^{t-1} p_r^s - x_i^t\right)}_{<0} < 0.$$

Hence $\tilde{x}_i^t = 0 \ \forall \ t$ is strictly dominant for all players $i = 1, \ldots, n$ because it results in $\sum_{t=1}^{T-1}(v \cdot \tilde{p}_i^t \prod_{s=1}^{t-1} p_r^s - \tilde{x}_i^t) = 0$, where $\tilde{p}_i^t = \frac{\tilde{x}_i^t}{\sum_{j\neq i}^n x_j^t + \tilde{x}_i^t + r}$ and $\tilde{p}_r^t = \frac{r}{\sum_{j\neq i} x_j^t + r}$.

If $T = \infty$, then $\mathbf{x} = \mathbf{0}$ is the unique Nash equilibrium.

It $T < \infty$, with $\tilde{p}_r^t = \frac{r}{\sum_{j\neq i} x_j^t + r} > \frac{r}{\sum_{j=1}^n x_j^t + r} = p_r^t$ for all $t < T$ we have

$$\prod_{t=1}^{T-1} p_r^t \left(p_i^T \cdot v - x_i^T\right) \leq \prod_{t=1}^{T-1} \tilde{p}_r^t \left(p_i^T \cdot v - x_i^T\right)$$

for any x_i^T with $p_i^T \cdot v \geq x_i^T$. If $x_i^t = 0$ for all $t = 1, \ldots, T-1$ and $i = 1, \ldots n$ we have $E[u_i(\mathbf{x}_i, \mathbf{x}_{-i})] = v \frac{x_i^T}{\sum_{j=1}^n x_j^T} - x_i^T$ with the well known solution $x_i^T = \frac{n-1}{n^2} v$. □

Proof of Lemma 1:

Proof : We will prove the following statement: If $\mathbf{x} = (\mathbf{x_1}, \ldots, \mathbf{x_n})$ with $\mathbf{x_i} = (x_i^1, \ldots, x_i^T)$ for $i = 1, \ldots, n$ satisfies inequality (1) for all periods $t = 1, \ldots, T$, then $\mathbf{x}_i = \mathbf{x}_j$ for all $i, j \in \{1, \ldots, n\}$.

If $T < \infty$, suppose that $x_i^T > x_j^T$ for some $i, j \in \{1, \ldots, n\}$, which implies $p_i^T > p_j^T$. With $\frac{p_i^T}{x_i^T} = \frac{p_j^T}{x_j^T}$ equation (1) implies $\frac{p_i^T}{x_i^T} \prod_{s=1}^{T-1} p_r^s \left(1 - p_i^T\right) < \frac{p_j^T}{x_j^T} \prod_{s=1}^{T-1} p_r^s \left(1 - p_j^T\right) \leq \frac{1}{v}$, which implies $x_i^T = 0$, a contradiction to $x_i^T > x_j^T$. Therefore $x_i^T = x_j^T$. Suppose now $\mathbf{x_i} \neq \mathbf{x_j}$ for some $i, j \in \{1, \ldots, n\}$ such that

$$\sum_{k=t}^{T} p_i^k \prod_{s=1}^{k-1} p_r^s > \sum_{k=t}^{T} p_j^k \prod_{s=1}^{k-1} p_r^s$$

for some $t = 1, \ldots, T-1$. Then inequality (1) in period t must be strict for any player i such that $x_i^t = 0$ and $p_i^t = 0$. Hence, the following chain of (in-)equalities holds:

$$\sum_{k=t+1}^{T} p_i^k \prod_{s=1}^{k-1} p_r^s = \sum_{k=t}^{T} p_i^k \prod_{s=1}^{k-1} p_r^s > \sum_{k=t}^{T} p_j^k \prod_{s=1}^{k-1} p_r^s \geq \sum_{k=t+1}^{T} p_j^k \prod_{s=1}^{k-1} p_r^s,$$

which directly implies that $x_i^{t+1} = 0$. By induction we have $x_i^s = 0$ for all $s \geq t$. But then $\sum_{k=t}^{T} p_i^k \prod_{s=1}^{k-1} p_r^s = 0$, a contradiction. Therefore we must have $\sum_{k=t}^{T} p_i^k \prod_{s=1}^{k-1} p_r^s = \sum_{k=t}^{T} p_j^k \prod_{s=1}^{k-1} p_r^s$ for all players $i, j = 1, \ldots, n$ and periods $t = 1, \ldots, T$. Comparing this equation for t and $t+1$ directly implies that $p_i^t = p_j^t$ for all $t = 1, \ldots, T$, which can only be satisfied if $x_i^t = x_j^t$ for all periods.

For the infinite case the proof can be adopted accordingly by setting $T = \infty$ and ignoring the paragraphs dealing with the last period. □

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